Acc. Sc. Torino - Atti Sc. Fis.
134 (2000), 175-190, 2 ff.

Elastic membrane behaviour of thin plates as wave propagation

Nota del Socio corrispondente ETTORE ANTONA
presentata nell'adunanza del 18 marzo 2000

Abstract. *Any membrane elastic dynamic phenomenon in a two-dimensional, continuous, not bounded, homogeneous and isotropic medium, without external field forces, has the nature of circular wave propagations. There are only two kinds of circular waves with different propagation speeds. At each time, it is always possible to separate the motion in two components, propagating respectively in two kinds of waves. The motion of a point, at a given time, depends on the motion of two kinds of waves at a previous time on two circumferences having as radius the distance travelled respectively by two circular waves.*

Riassunto. *Ogni fenomeno dinamico elastico membranale, in un mezzo bidimensionale, continuo, non limitato, omogeneo e isotropo, in assenza di forze esterne, ha la natura di propagazioni per onde circolari. Ci sono solo due tipi di onde circolari aventi differenti velocità di propagazione. In ogni istante è sempre possibile suddividere lo stato di moto in due componenti propagantisi rispettivamente nei due tipi di onde. Il moto in un punto a un certo tempo dipende dalle medie dei moti dei due tipi di onde a un tempo precedente su due circonferenze aventi raggi pari alle distanze coperte dalle due onde circolari.*

1. Mathematical model

1.1 General remarks

It is well known the possibility of considering any three-dimensional dynamic phenomenon in continuous and homogeneous media (without external field forces) as a wave propagation (see, for example, [1]).

In order to discuss the problem of longitudinal dynamic phenomenon of one-dimensional media, with time variable boundary condition location, the A., ([2]), highlighted the benefit, in order to develop an appropriate mathematical model, to make use of a vision of a dynamic phenomenon as a sum of wave propagations.

By use of such a vision, the A. considered then various problems concerning dispersive media, bending phenomena, and phenomena of vibrating strings with both uniform and variable tension ([3], [4], [5], [6]).

In the case of three-dimensional media, the basic idea of wave propagations becomes useful also to try discretized approaches to the problems consequent upon time variable boundary conditions locations.

The present work deals with the problem of the membrane dynamic behaviour of flat shells, by the above mentioned point of view.

An approximate, mathematical model for the problem of the membrane dynamic behaviour of flat panels can be derived from three-dimensional model (see, for instance ,[1]) after introducing the right hypoteses and corrections. The hypotheses of *"plane stress"* seem to be the most appropriate, since we consider the case where the transverse external loads are zero.

This behaviour implies, if we consider the Poisson's effect, that the internal transverse displacements are not zero, but assuming in the model the hypoteses of small thickness, at limit vanishing, they can be neglected, by the point of view both of internal strain-stress and of kinetic energy.

The hypotheses to be adopted, for the above prospected case, are the following:

- membrane stress are constant along the thickness,
- displacements and strains in the middle plane are constant along the thickness,
- other displacements, strain and stress are zero.

Among other things, a consequence of such hypotheses is a necessary correction of expression of a Lamè's constant in terms of longitudinal elastic modulus E and Poisson's constant χ, in comparison with three-dimensional problem formulation. Under the above mentioned hypotheses we can write, for a generic orthogonal reference sistem x, y in the middle surface plane[1]:

$$\varepsilon_x = \frac{(1+\chi)\sigma_x - \chi 2\sigma_m}{E},$$

$$\varepsilon_y = \frac{(1+\chi)\sigma_y - \chi 2\sigma_m}{E}, \quad 1a,b,c)$$

$$\gamma_{xy} = \frac{2(1+\chi)}{E}\tau.$$

[1] Eq.s 1a, b, c) are obtained from relations refered to principal axis and operating an axe rotation.

where $\varepsilon_x, \varepsilon_y$ and γ_{xy}[2] are the local strains and σ_x, σ_y and τ_{xy}[3] are the local stresses.

In 1a,b,c) it is

$$2\sigma_m = \sigma_x + \sigma_y . \qquad 2)$$

Dilatation γ is expressed by divergence, which, taking into account (2), equals:

$$\gamma = \varepsilon_x + \varepsilon_y = \frac{(1+\chi)-2\chi}{E}2\sigma_m = \frac{1-\chi}{E}2\sigma_m ,$$

from which we obtain:

$$2\sigma_m = \frac{E}{1-\chi}\gamma . \qquad 3)$$

Taking into account (3), equations 1a,b,c) become:

$$\varepsilon_x = \frac{(1+\chi)\sigma_x - \chi\frac{E}{1-\chi}\gamma}{E},$$

$$\varepsilon_y = \frac{(1+\chi)\sigma_y - \chi\frac{E}{1-\chi}\gamma}{E}, \qquad 1'a,b,c)$$

$$\gamma_{xy} = \frac{2(1+\chi)}{E}\tau .$$

From the above equation we can compute σ_x, σ_y and τ:

[2] Tangenzial strais is here defined $\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$, where u_x and u_y an the displacement components.

[3] Normal stresses are positive in tension.

$$\sigma_x = \frac{E}{1+\chi}\varepsilon_x + \frac{\chi E}{(1+\chi)(1-\chi)}\gamma,$$
$$\sigma_y = \frac{E}{1+\chi}\varepsilon_y + \frac{\chi E}{(1+\chi)(1-\chi)}\gamma, \quad 4a,b,c)$$
$$\tau = \frac{E}{2(1+\chi)}\gamma.$$

Introducing the following expression of Lamè's constants for the membrane two-dimensional case:

$$\lambda = \frac{\chi E}{(1+\chi)(1-\chi)}, \quad \mu = \frac{E}{2(1+\chi)},$$

equations 4a,b,c) give the following equations of stress state under the hypotheses of *"plane stress"*:

$$\sigma_x = \lambda\gamma + 2\mu\varepsilon_x,$$
$$\sigma_y = \lambda\gamma + 2\mu\varepsilon_y, \quad 5a,b,c)$$
$$\tau = \mu\gamma_{xy}.$$

No comment is necessary about identification between μ and shear modulus G, whose expression in terms of Young's modulus E and χ has been already introduced in (1c).

1.2 Dynamic equations

Let x,y be a rectangular coordinate system in the (middle) plane of panel and r,ω the corresponding polar coordinate system. Let u,v be the component of vector **s** in the first system, so that:

$$\boldsymbol{s}=\boldsymbol{i}u+\boldsymbol{j}v.$$

Let f be a scalar. The following usual expressions are valid:

$$div\,\boldsymbol{s} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \quad 6)$$

$$\Delta \boldsymbol{s} = \boldsymbol{i}\Delta u + \boldsymbol{j}\Delta v = \boldsymbol{i}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \boldsymbol{j}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \quad 7)$$

$$grad\, f = \boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y}, \qquad 8)$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = div\, grad\, f, \qquad 9)$$

$$rot\, \boldsymbol{s} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{k}, \qquad 10)$$

$$grad\, div\, \boldsymbol{s} = \boldsymbol{i}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + \boldsymbol{j}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right). \qquad 11)$$

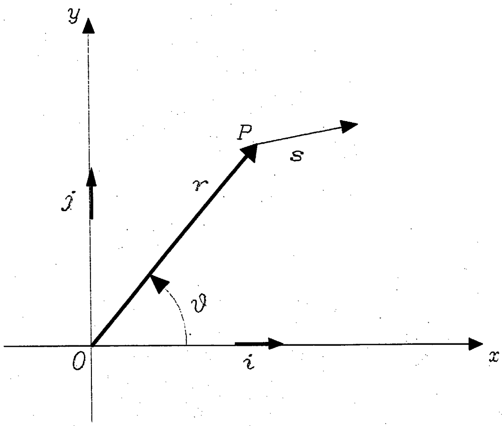

Figure 1

Let us assume that:

$$\boldsymbol{s} = (s_x, s_y), \; \gamma = \frac{\partial s_x}{\partial x} + \frac{\partial s_y}{\partial y} \qquad 12a,b)$$

and, taking into account 5a,b,c), and (9), the scalar dynamic equations are:

$$\begin{cases} (\lambda+\mu)\dfrac{\partial\gamma}{\partial x}+\mu\Delta s_x+\rho F_x=\rho\ddot{s}_x, \\ \qquad 13a,b) \\ (\lambda+\mu)\dfrac{\partial\gamma}{\partial y}+\mu\Delta s_y+\rho F_y=\rho\ddot{s}_y, \end{cases}$$

where F_x and F_y are the components of possible external loads.

Eqs. 13a,b) can be summarized in the following scalar equation:

$$(\lambda+\mu)\, grad\, div\, \mathbf{s}+\mu\,\Delta\mathbf{s}+\rho\mathbf{F}=\rho\mathbf{s} \quad . \qquad 14)$$

1.3 Irrotational oscillations

If we assume that:

$$rot\mathbf{s}=0,$$

it is well known that there exists a function φ such that:

$$\mathbf{s}=grad\,\varphi.$$

Eq. (14) , with $\mathbf{F}=0$, yields:

$$(\lambda+\mu)\Delta\mathbf{s}+\mu\,\Delta\mathbf{s}=\rho\,\ddot{\mathbf{s}}$$

and, moreover:

$$(\lambda+2\mu)\Delta s=\rho\,\ddot{s}. \qquad 16)$$

If we put:

$$b=\sqrt{\frac{\lambda+2\mu}{\rho}},$$

it follows that:

$$\Delta s-\frac{1}{b^2}\ddot{\mathbf{s}}=0,$$

which is equivalent to the following two scalar equations of the same form in s_x and s_y:

$$\begin{cases} \Delta s_x - \dfrac{1}{b^2}\ddot{s}_x = 0, \\ \Delta s_y - \dfrac{1}{b^2}\ddot{s}_y = 0, \end{cases} \qquad 17a,b)$$

and represents an extension of equation of vibrating strings to the two-dimensional case.

1.4 Zero dilatation or solenoidal oscillations

If we put $div\, \mathbf{s}=0$, we obtain from eq. (14), with $\boldsymbol{F}=0$:

$$\mu\, \Delta \mathbf{s} = \rho\, \ddot{\mathbf{s}}, \qquad 18)$$

that is, taken $a = \sqrt{\dfrac{\mu}{\rho}}$:

$$\Delta \mathbf{s} = \frac{1}{a^2}\, \ddot{\mathbf{s}}, \qquad 19)$$

which is equivalent to two scalar equations of the same form in s_x and s_y.

These equations belong to "vibrating strings" kinds.

1.5 Oscillations of any kind

The most general kind of motion (displacement **s**) can always be led back to a combination of two displacements, $\mathbf{s}_1$ and $\mathbf{s}_2$, irrotational the one and solenoidal the other. In fact we can always determine a function φ such that:

$$\Delta\varphi = div\, \mathbf{s}. \qquad 20)$$

From (9), taking into account (20), it follows that:

$$div(\mathbf{s} - grad\, \varphi) = 0, \qquad 21)$$

which means also that it is possible to find a vector Ω, such that:

$$\mathbf{s} - grad\, \varphi = rot\, \Omega \qquad 22)$$

Thus, on the basis of (21) and (22),

$$\mathbf{s}_1 = grad\, \varphi \quad , \quad \mathbf{s}_2 = rot \varphi \qquad 23a,b)$$

result, respectively, irrotational the one and irrotational the other, with

$$\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2\ . \qquad 24)$$

2. General properties of wave equation

2.1 Propagation by circular waves

As a consequence of eqs. 17a,b) and (19), let us investigate about the following kind of equation:

$$\Delta\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} \quad . \qquad 25)$$

Let us consider a circumference l having radius r and centre P and delimiting a circle A.

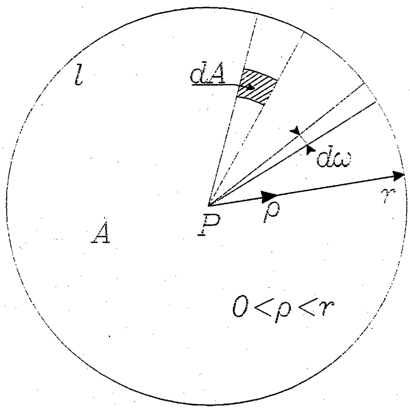

Figure 2

Let $\overline{\Psi}$ be the integral of Ψ at time t on the boundary l:

$$\overline{\Psi}(r,t) = \frac{1}{2\pi r^2}\int_l \Psi dl \quad . \qquad 26)$$

Making use of polar coordinates ρ, ω we have:

$$dl = r d\omega$$

and eq.(26) becomes:

$$\overline{\Psi}(r,t) = \frac{1}{2\pi r} \int_0^{2\pi} \Psi \, d\omega \; . \qquad 27)$$

Note that $r\overline{\Psi}(r,t) = \Psi$ coincides with the medium of Ψ on the circumference having radius *r*.

On the circle *A*, the following relation is valid:

$$\int_A \Delta\Psi \, dA = \frac{1}{c^2} \int_A \frac{\partial^2 \Psi}{\partial t^2} dA \; . \qquad 28)$$

Manipulating the left side member in eq.(28), noting that *dA=dl dr*, it follows that:

$$\int_l \frac{d\Psi}{dn} dl = \frac{1}{c^2} \int_A \frac{\partial^2 \Psi}{\partial t^2} dA \; . \qquad 29)$$

On the other hand, it is $\frac{d}{dn} = \frac{\partial}{\partial r}$, $dA = \rho \, d\omega \, \rho$, where $0 < \rho < r$ and $dl = r \, d\omega$, so that eq. (29) yields:

$$\int_l \frac{\partial \Psi}{\partial r} r \, d\omega = \frac{1}{c^2} \int_0^r \int_0^{2\pi} \frac{\partial^2 \Psi}{\partial t^2} \rho \, d\omega \, d\rho \; , \qquad 30)$$

$$r \int_0^{2\pi} \frac{\partial \Psi}{\partial r} d\omega = \frac{1}{c^2} \int_0^r \rho \, d\rho \int_0^{2\pi} \frac{\partial^2 \Psi}{\partial t^2} d\omega \; , \qquad 31)$$

Finally, first derivation of (27) and manipulation of the result, taking into account (31), yield:

$$r^2 \frac{\partial \overline{\Psi}}{\partial r} = \frac{1}{c^2} \int_0^r \rho^2 \frac{\partial^2 \overline{\Psi}}{\partial t^2} d\rho \; . \qquad 32)$$

Deriving eq. (32) with respect to *r* , it follows:

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial \overline{\Psi}}{\partial r} \right) = \frac{1}{c^2} r^2 \frac{\partial^2 \overline{\Psi}}{\partial t^2} \qquad 33)$$

and dividing (33) by r^2 we obtain:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \overline{\Psi}}{\partial r} \right) = \frac{1}{c^2} \frac{\partial^2 \overline{\Psi}}{\partial t^2} \; , \qquad 34)$$

where $\overline{\Psi} = \dfrac{1}{2\pi\, r^2} \int_l \Psi\, dl$.

Such a result can be demonstrated in another way (see for instance App. A). The left hand member in eq. (34) can be rewritten in the form[4]:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \overline{\Psi}}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\overline{\Psi}\right), \qquad 35)$$

so that eq.(34) itself becomes:

$$\frac{\partial^2\left(r\overline{\Psi}\right)}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2\left(r\overline{\Psi}\right)}{\partial t^2}, \qquad 36)$$

which is the equation of vibrating strings for the function $r\overline{\Psi}$, with variables r,t.

2.2 Propagation in two-dimensional problems

Because the fact that travelling waves are not zero in $r=0$, it is not possible to obtain, for the two-dimensional case, an analogous equation to Poisson's one for the three-dimensional[5] case, which yields the general solution of equation

$$\Delta\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}, \qquad 25rep)$$

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[4] In fact, it is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\overline{\Psi}}{\partial r}\right) = \frac{1}{r^2}\left[2r\frac{\partial\overline{\Psi}}{\partial r} + r^2\frac{\partial^2\overline{\Psi}}{\partial r^2}\right] = \frac{2}{r}\frac{\partial\overline{\Psi}}{\partial r} + \frac{\partial^2\overline{\Psi}}{\partial r^2}$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\overline{\Psi}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[\overline{\Psi} + r\frac{\partial\overline{\Psi}}{\partial r}\right] = \frac{1}{r}\frac{\partial\overline{\Psi}}{\partial r} + \frac{\partial^2\overline{\Psi}}{\partial r^2} + \frac{1}{r}\frac{\partial\overline{\Psi}}{\partial r} = \frac{2}{r}\frac{\partial\overline{\Psi}}{\partial r} + \frac{\partial^2\overline{\Psi}}{\partial r^2}$$

[5] See the Poission's formula as obtained in [1] and as taken up again and used in [7].

when it is defined in a not bounded (two-dimensional) space and when initial values Ψ_0, $\dot{\Psi}_0$ and $\frac{\partial \Psi}{\partial \tau}$ are established wherever in the space[6].

The general integral can be put in the form:

$$r\overline{\Psi} = f(r+ct) + g(r-ct). \qquad 37)$$

Function $r\overline{\Psi}$ is, really, the average along the circumference with radius r, at time t. Denoting it with $\hat{\Psi}(r,t)$, it is possible to make use of the considerations done, [8], about one-dimensional propagation phenomena:

$$f(ct) = -g(-ct).$$

Note that, when $r \to 0$, $\hat{\Psi} \to \Psi(r=0)$. Equation 37) gives values of $r\overline{\Psi} = \hat{\Psi}$ that are travelling in the increasing r direction under the component g. The same f and g can be considered as travelling in the plane where r's are negative starting from the origin in the decreasing r modulus direction and in the increasing r modulus direction respectively.

In other words, f and g can be considered as waves trapassing the point $r=0$ from the negative r conception to the positive r conception and vice-versa respectively. Such circumstance is true also in the three-dimensional case, where it is not utilized because of previously mentioned Poisson's theory.

If $\hat{\Psi}(r,t^*)$ and $\dot{\hat{\Psi}}(r,t^*)$ are given, where t^* is an assigned time value, it is possible to determine the two travelling waves $t(r+ct)$ and $g(r-ct)$, which the phenomenon can be decomposed in.

We can write:

$$\hat{\Psi}(r,t) = f(r+ct) + g(r-ct) = f(q) + g(p), \qquad 38)$$

where $q = r + ct$, $p = r - ct$. Therefore:

$$\dot{\hat{\Psi}}(r,t) = c\frac{df}{dq} - c\frac{dg}{dp} \qquad 39)$$

and, in particular, as time t^* is concerned:

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[6]In fact, it is impossible in this case to determine a radially propagating (medium) function having zero value at $r=0$.

$$\dot{\hat{\Psi}}(r,t^*) = c\left(\frac{df}{dq}\right)_{r,t^*} - c\left(\frac{dg}{dp}\right)_{r,t^*} . \qquad 39')$$

Let us now consider the representations of $f(r+ct^*)$ and $g(r-ct^*)$, such that:

$$A = g(r - dr - ct^*),$$
$$B = f(r + dr + ct^*).$$

From eq. 38) we have:

$$\hat{\Psi}(r,t^*) = f\left[r + dr + c\left(t^* - \frac{dr}{c}\right)\right] + g\left[r - dr - c\left(t^* - \frac{dr}{c}\right)\right],$$

where the displacement in r at t^* is obtained from values of f and g in an appropriately changed position and time. Similarly, we have:

$$\hat{\Psi}\left(r, t^* + \frac{dr}{c}\right) = f\left[r + dr + ct^*\right] + g\left[r - dr - ct^*\right]. \qquad 40)$$

Putting now the speed $\dot{\hat{\Psi}}(x,t)$ under the form:

$$\dot{\hat{\Psi}}(r,t) = \left[\hat{\Psi}\left(r, t^* + \frac{dr}{c}\right) - \hat{\Psi}(r,t^*)\right]\frac{c}{dr},$$

we obtain:

$$\dot{\hat{\Psi}}(r,t) = \frac{c}{dt}\left[f(r + dr + ct^*) - f(r + ct^*) + g(r - dr - ct^*) - g(r - ct^*)\right]$$

or

$$\dot{\hat{\Psi}}(r,t^*) = c\left\{\left(\frac{\partial f}{\partial q}\right)_{r,t^*} - \left(\frac{\partial g}{\partial p}\right)_{r,t^*}\right\}, \qquad 39'')$$

which is another form for the meaning of eq. (39).

Deriving eq. (38) with respect to r and putting eq.(*39″*) in another form, we obtain:

$$\frac{\partial \hat{\Psi}}{\partial r}(r,t) = \frac{\partial f}{\partial q}(r + ct) + \frac{\partial g}{\partial p}(r - ct), \qquad 41)$$

$$\frac{1}{c}\frac{\partial \hat{\Psi}}{\partial t}(r,t)=\frac{\partial f}{\partial q}(r+ct)-\frac{\partial g}{\partial p}(r-ct) \qquad 42)$$

or, in other words:

$$\hat{\Psi}_r(r,t)=f_q(r+ct)+g_p(r-ct), \qquad 41')$$

$$\frac{1}{c}\dot{\hat{\Psi}}(r,t)=f_q(r+ct)-g_p(r-ct). \qquad 42')$$

If we put:

$$\hat{\Psi}=\hat{\Psi}(r,t^*),\ f=f(r+ct^*),\ g=g(r-ct^*),\ f_x=\left(\frac{\partial f}{\partial q}\right)_{r,t^*},$$

$g_r=\left(\frac{\partial g}{\partial p}\right)_{r,t^*}$, from (38) and $(39'')$ we can obtain:

$$\hat{\Psi}=f+g\ ,\quad g=\hat{\Psi}-f,$$

$$\frac{\dot{\hat{\Psi}}}{c}=\frac{\partial f}{\partial q}-\frac{\partial g}{\partial p}=f_r-\hat{\Psi}+f_r=2f_r-\hat{\Psi}.$$

Therefore we have:

$$f_r=\frac{\hat{\Psi}+\dot{\hat{\Psi}}/c}{2}, \qquad 43a)$$

$$g_r=\frac{\hat{\Psi}-\dot{\hat{\Psi}}/c}{2}. \qquad 43b)$$

Eq.s (43a) and (43b) can be integrated as follows:

$$f = \frac{\hat{\Psi}}{2} + \int \frac{1}{2} \frac{\dot{\hat{\Psi}}}{c} dr + c_1 \ ,$$

$$g = \frac{\hat{\Psi}}{2} - \int \frac{1}{2} \frac{\dot{\hat{\Psi}}}{c} dr + c_2 \ ,$$

where it results $c_2 = -c_1$, because of eq.(38).
On the other hand, $c_2 = -c_1$ is arbitrary but indifferent because $f = c_1$, $g = -c_1$ generate $\hat{\Psi} = 0$ and $\dot{\hat{\Psi}} = 0$. Therefore it is possible to choose $c_1 = 0$. As a result we obtain:

$$f(r + ct^*) = \frac{\hat{\Psi}}{2} + \int \frac{\dot{\hat{\Psi}}}{2c} dr, \qquad 44a)$$

$$g(r - ct^*) = \frac{\hat{\Psi}}{2} - \int \frac{\dot{\hat{\Psi}}}{2c} dr. \qquad 44b)$$

Eq.s (44a) and (44b) allow us to obtain f and g from $\hat{\Psi}$ and $\dot{\hat{\Psi}}$. When $\hat{\Psi}(r,t^*)$ and $\dot{\hat{\Psi}}(r,t^*)$ are known, it is possible to calculate $\int \dot{\hat{\Psi}} dr$, starting from a value $\bar{r}$ of r, and to determine f and g by means of (44a) and (44b).
Such functions $f(r + ct)$ and $g(r - ct)$ are travelling waves.

Conclusions

Similarly to what happens in the three dimensional combinuous dynamic, any membrane elastic dynamic phenomenon in a two-dimensional, continuous, not bounded, homogeneous and isotropic medium, without external field forced, can be reduced to circular wave propagations. There are only two kinds of circular waves with different propagation speeds. At each time, it is always possible to separate the motion in two components, propagating respectively in two kinds of waves. The motion of a point, at a given time, depends on the motion of two kinds of waves at a previous time on two circumferences having as radius the distance travelled respectively by two circular waves.

Appendix A

We require that for an unknown function $\bar{f}$ holds the following relation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \bar{f}}{\partial r}\right)=\frac{1}{c^2}\frac{\partial^2 \bar{f}}{\partial t^2}r^k \qquad A1)$$

where k is chosen in order function f could verify the wave equation.

If we multiply eq.(B1) for r^2 and integrate with respect to r, we obtain:

$$r^2\frac{\partial \bar{f}}{\partial r}=\frac{1}{c^2}\int r^2\frac{\partial^2 \bar{f}}{\partial t^2}r^k\, dr \qquad A2)$$

Let us suppose now that $\bar{\varphi}=r^{\mu}\int\varphi\, dl=r^{\mu+1}\int\varphi\, d\omega$, with μ unknown. Eq.(B2) than becomes (taking into account that $dA=r\, d\omega\, dr$):

$$r^2r^{\mu+1}\int\frac{\partial f}{\partial r}d\omega=\frac{1}{c^2}\int r^k r^2 r^{\mu+1}dr\int\frac{\partial^2 f}{\partial t^2}d\omega$$

and, moreover:

$$r^2r^{\mu}\int\frac{\partial f}{\partial r}dl=\frac{1}{c^2}\int_A r^k r^2 r^{\mu}\frac{\partial^2 f}{\partial t^2}dA \qquad B3)$$

If the wave equation is satisfied, on the basis of eq.(29), exists a function Ψ such that:

$$\int\frac{\partial\Psi}{\partial r}r\, d\omega=\frac{1}{c^2}\int_A\frac{\partial^2\Psi}{\partial t^2}dA \qquad B4)$$

From (B3) and (B4), if we impose $r^2r^{\mu+1}=r$, it follows that $2+\mu+1=1$, that is $\mu=-2$ and $f=\Psi$, obtaining, finally:

$$\int\frac{\partial\Psi}{\partial r}r\, d\omega=\frac{1}{c^2}\int_A\frac{\partial^2\Psi}{\partial t^2}r^{k+2+\mu}dA$$

From eq. (B3) it follows that, if $k+2+\mu=0$ (which implies, if $\mu=-2$, that $k=0$), the wave equation is valid.

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