Acc. Sc. Torino - Atti Sc.
135 (2001),135-161, 2 ff., 4 tabb.

MECCANICA DEI CONTINUI

Some considerations about the concepts of asymptotic approach

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presentata nell'adunanza del 7 Marzo 2001

Abstract. *The first part of this paper concerns the introduction to the basics of asymptotic approach, with reference to the work of Prof. Cicala. The explanation is supported by means of an original application. The aim is to find an approximate solution to the structural beam problem, emphasizing the possibility of obtaining different models with respect to the initial assumptions as discussed in the second part. The power of Cicala's method lies both in the coherence of its initial positions and in the derivation of the approximation. The right procedure and the main characteristics of this method are carefully described. The differences between this procedure and an alternative one are also pointed out. The method introduces the order of smallness of every term and, after considering specific observations, it obtains the order of magnitude of the variables involved in connection to the specific asymptotic assumption. Moreover rearranging the approximate solution equations it is possible to take into account nonlinear terms if necessary. Successive approximations are calculated in an original scheme that reduces the system to a "scalarized form" according to the classifications of the unknowns. This form provides the starting point for a specific connection to general mathematical theory as the third part of this paper is meant to show in an original way. Further considerations*

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are introduced in order to ascertain whether the subsequent approximation is convenient or not. Thanks to this derivation, Cicala's work received a new and previously unidentified merit.

Sommario. *Nella prima parte di questo lavoro si presentano i concetti base degli approcci asintotici facendo riferimento al lavoro del Cicala ed appoggiandosi ad una applicazione originale. Nella seconda parte si ricercano soluzioni approssimate al problema strutturale della trave evidenziando la possibilità di ottenere modelli differenti in funzione delle assunzioni iniziali. Si sottolineano le potenzialità del metodo di Cicala sia per la coerenza delle posizioni iniziali sia nella derivazione dell'approssimazione, descrivendone la corretta procedura e le principali caratteristiche. Vengono inoltre puntualizzate le differenze in riferimento ad una metodologia alternativa. Il metodo introduce l'ordine di infinitesimo di ciascun termine in base a particolari osservazioni ed ottiene l'ordine di grandezza delle variabili in riferimento ad specifiche assunzioni asintotiche. Riorganizzando le equazioni della soluzione approssimata è possibile introdurre termini non lineari se necessario. Successive approssimazioni vengono determinate in uno schema originale che riduce il sistema in una forma "scalarizzata" in funzione della classificazione delle incognite. Essa diventa il punto di partenza di uno specifico collegamento con teorie matematiche generali come mostrato in modo originale nella terza parte. Ulteriori considerazioni vengono introdotte per stabilire la convenienza della approssimazione successiva. Questa derivazione permette quindi di attribuire al lavoro di Cicala valenze nuove e fino ad ora non identificate.*

1 General remarks on asymptotic approaches

Asymptotic approaches are applicable to all problems concerned with the computation of approximate solutions, when some of the quantities, important to set out the problem, can be considered small with respect to the remaining ones (for instance the thickness in the shells or the section dimensions in the beams). This approach, which could be ordained to establish the basic philosophy of many issues in engineering science and in mathematical physics, is the opposite of the axiomatic approach in which the introduction of specific (often ingenious) assumptions imposes constraints to the nature of the solutions. In the author's opinion any attempt that in some respects refers to the concept of asymptotic approach, has to be compared to Cicala's formulation. The rigor and profundity of his statement can provide a useful touchstone especially for the following peculiar characteristics:

- the order of smallness of the unknowns (quantities or variables) are evaluated in advance,
- the effect of the differentiation is taken into account in the orders of smallness, with regard to the nature of the unknown function and the asymptotic behavior,
- on this basis, the knowledge of the effective order of smallness of each terms in each equation is obtained,
- the asymptotic approach is applied also to the boundary conditions, within the problem of determining the constants in the case of both homogeneous and not homogeneous problems.

As far as asymptotic approaches are concerned, Placido Cicala gave a great contribution proposing a theory where not only all the axiomatic approaches appear to be particular cases, but they are also understood in their limits and implications. The results obtained by St.Venant, Timoshenko, Kirchhoff, Mindlin and others are overhang at a logical and mathematical level, by means of a vision that is based on the subsequent points: a) including the problem at issue in a family obtained from the reduction to zero of a fundamental parameter, b) thinking about the solution and the equation with all its terms (and the contribution of the unknown) as the expansion of a series of functions that constitute a complete basis in the space of coordinates in which some dimensions are small with respect to the others, and considering the order of smallness of each component with respect to the fundamental parameter (it is worth noting that the tridimensional equations of the continuum become infinite in number and with infinite terms in bidimensional or unidimensional spaces for the shells and beams respectively), c) decomposing the systems of infinite equations in reduced systems with a finite number of terms and up to a specific order of smallness while neglecting the others. For each of those steps, Placido Cicala created original procedures in which some surprising aspects can be found : the possibility of evaluating the order of smallness of the unknowns before solving the system, the semi convergent [2] nature of the series obtained in certain refining processes of the solution and so on. The asymptotic approach can be favorably used in all the problems that have an analytic formulation maintaining its significance whereas an intrinsic parameter becomes zero. If we compare it with the axiomatic approach, in which the simplifications based on the smallness of a parameter are introduced heuristicly in the form of "a priori" assumptions, the following advantages arise: the approximation level is not imposed by certain axioms on the solutions, a sequence of approximating steps of increasing level, is determined and the axiomatic models are emphasized; each step contributes towards a coherent connection between the loads and the boundary conditions. The problem of a St. Venant beam over an elastic foundation can be considered as an example of some aspects of the asymptotic approach solution [9],[3]. The situation is depicted in fig.1 where the symbols have the usual meaning. The resulting equation, founded on the axiomatic behavior of the transversal sections, is as follows:

$$(EIy'')'' - Hy'' + ky = p, \qquad (0 \leq x < l). \tag{1}$$

The asymptotic approach is applied to this equation and it is restricted to the choice of the most important terms characteristic of certain classes of solutions consistent with the axiom. Starting from an equation defined in a unidimensional space and deduced from an axiom, is not the right way to reduce the tridimensional problem to the unidimensional one. However the reported example is very sig-

[2]in the sense of asymptotic series or not convergent series (see [7])

nificant. Considering that the problem derives from a family where a parameter δ is going to zero asymptotically (δ multiplies the thickness of the beam and it is considered as principal infinitesimal), it is interesting to establish a classification of the importance of the terms in the equation if considered to the asymptote. The following assumptions are introduced: $H = o(\delta^0), k = o(\delta^0), I = o(\delta^2)$, that is H and k remain constant as the δ parameter tends to zero while I is an infinitesimal of order 2 (in the definition of the I parameter, the section A is maintained constant as the thickness tends to zero). Such assumptions are applied to the homogeneous case:

$$(EIy'')'' - Hy'' + ky = 0. \tag{2}$$

It is important to introduce the effect of the derivation in the order of smallness of each term. This is not a mere dimensional analysis that fixes the order of smallness of the derivative with respect to the function once and for all (usually to the zero value). That value is considered into the procedure in order to define the different classes of solution. An expression as the following one is to be introduced:

$$y' = o(y\delta^{-g}). \tag{3}$$

Each term α in the previous equation is in the form :

$$\frac{\alpha}{y} = o(\delta^m \delta^{-gn}), \tag{4}$$

where m is the order of smallness of the coefficient and n is the order of the derivative in the α term. Fig.2 shows in the m, n plane, the representative points of the three terms of the equation. It is possible to determine the g values that give specific approximations to the problem, neglecting the smaller terms in the equation. The order of smallness of the i term in the equation is such that: $i = m - gn$. The straight line $i = constant$ contains the terms of the same order with g equal to its slope. The terms positioned under the line are more important than those on the line while the terms above the line are less important. The inferior polygonal envelope of the points in the m, n plane is made up of part of straight lines each of them representative of a g value consistent with the problem and containing the most important terms. Given $g = 0$ and $g = 1$ the solutions of the reduced problem are respectively:

$$y_1 = A_1 e^{x\sqrt{\frac{k}{H}}} + B_1 e^{-x\sqrt{\frac{k}{H}}}; \quad y_2 = A_2 e^{x\sqrt{\frac{H}{EI}}} + B_2 e^{-x\sqrt{\frac{H}{EI}}}. \quad (5)$$

The second equation is to be considered with the exception of a rigid translation eliminable with the specific boundary conditions. The two approximate solutions display four constants, the same number as in the determination of the general integral of the complete fourth order equation. The general integral can be approximated by setting :$y = y_1 + y_2$. The same result is obtained through a series expansion of the second degree algebraic equation derived from the determinantal equation associated to the general homogeneous equation when $y = Ce^{\lambda x}$ is introduced. Indeed the determinantal equation obtained is :

$$EJ\lambda^4 - H\lambda^2 + K = 0, \quad (6)$$

with solutions:

$$\lambda = \pm\sqrt{\frac{H}{2EJ}\left(1 \pm \sqrt{1 - \frac{4K}{H}\frac{EJ}{H}}\right)}. \quad (7)$$

Introducing the order of smallness of the coefficient as assumed previously, we get the following simplification:

$$\lambda^2 \approx \frac{H}{2EJ}\left[1 \pm \left(1 - \frac{2K}{H}\frac{EJ}{H}\right)\right] = \left\{ \begin{array}{l} K/H \\ H/EJ \end{array} \right. . \quad (8)$$

Some points regarding the boundary conditions need to be highlighted, taking into consideration the not homogeneous term p. The general homogeneous boundary conditions are:

$$y_0 = 0, \quad y_l = 0, \quad M_0 = 0, \quad M_l = 0. \quad (9)$$

If the $\bar{y} = \bar{y}(x)$ is the particular integral of the initial equation, the approximate expression for the general integral is:

$$y = \bar{y} + A_1 e^{x\sqrt{\frac{k}{H}}} + B_1 e^{-x\sqrt{\frac{k}{H}}} + A_2 e^{x\sqrt{\frac{H}{EI}}} + B_2 e^{-x\sqrt{\frac{H}{EI}}}. \qquad (10)$$

We can satisfy the boundary conditions by using the asymptotic approach. Table tab.1 is obtained, evaluating the order of smallness of the four quantities that are important in the definition of the boundary conditions, i.e.: the displacement y, the derivative y', the external bending moment M, and the external transverse load R,

Tab.1

		y	y'	M	R
y_1	expression	y_1	y_1'	EIy_1''	$Hy_1' + \ldots EIy_1'''$
	o.s.	y_1	y_1	$\delta^2 y_1$	y_1
y_2	expression	y_2	y_2'	EIy_2''	$Hy_2' + EIy_2'''$
	o.s.	y_2	$\delta^{-1} y_2$	y_2	$\delta^{-1} y_2$

The boundary conditions system at $x = 0, l$ is as follows:

$$\begin{aligned} y(0) &= 0, & y_1(0) + y_2(0) &= -\bar{y}(0), \\ y(l) &= 0, & y_1(l) + y_2(l) &= -\bar{y}(l), \\ M(0) &= 0, & y_1''(0) + y_2''(0) &= 0, \\ M(l) &= 0, & y_1''(l) + y_2''(l) &= 0. \end{aligned} \qquad (11)$$

The determination of the order of smallness of the terms contained in the approximate solutions y_1, y_2 gives the possibility to identify the equal order terms in the approximate equations concerning the boundary conditions. Those orders are as follows:

$$\begin{aligned} \lim_{x\to 0} e^{x\sqrt{k/H}} &= o(\delta^0), & \lim_{x\to l} e^{x\sqrt{k/H}} &= o(\delta^0), \\ \lim_{x\to 0} e^{-x\sqrt{k/H}} &= o(\delta^0), & \lim_{x\to l} e^{-x\sqrt{k/H}} &= o(\delta^0), \\ \lim_{x\to 0} e^{x\sqrt{H/EI}} &= o(\delta^0), & \lim_{x\to l} e^{x\sqrt{H/EI}} &= o(\delta^0) e^{l\sqrt{H/EI}}, \\ \lim_{x\to 0} e^{-x\sqrt{H/EI}} &= o(\delta^0), & \lim_{x\to l} e^{-x\sqrt{H/EI}} &= o(\delta^\infty). \end{aligned} \qquad (12)$$

The algebraic system in the unknowns $A_1, B_1, A{*}_2, B_2$, produces the subsequent table (Tab.2) about the order of smallness of the coefficients. Finite values are considered for $\bar{y}(0), \bar{y}(l)$, null values for $\bar{y}''(0), \bar{y}''(l)$.

Tab.2

	A_1	B_1	$A{*}_2$	B_2	
$y = 0$	$\underline{0}$	$\underline{0}$	∞	0	$\underline{0}$
$y = l$	$\underline{0}$	$\underline{0}$	0	∞	$\underline{0}$
$M = 0$	$\underline{2}$	$\underline{2}$	∞	$\underline{0}$	∞
$M = l$	$\underline{2}$	$\underline{2}$	$\underline{0}$	∞	∞
	0	0	2	2	

In tab.2 $A{*}_2 = A_2 exp(l\sqrt{H/EI})$. The order of smallness of the unknowns is reported in the last line of Tab.2. It derives from the difference between the order of smallness of a numerator, determined substituting the known column in the corresponding column of the coefficients, and a denominator consisting of the determinant of the coefficients itself. The first approximation is indicated with a bar under the terms in the table. The first approximating solving system for the constants is decomposed in a simplified system with two equations and two decoupled equations. Better approximations could be obtained taking into consideration further terms other than the principal ones.

2 Derivation of the classical approximate beam models by using Cicala's approach

2.1 Introduction

On the basis of Cicala's work ([11],[14],[15],[16]), a well defined procedure comes out. It defines, in conjuction with the specific asymptotic approach, a certain approximate solution to the structural both bidimensional ([4]) and unidimensional problem. First

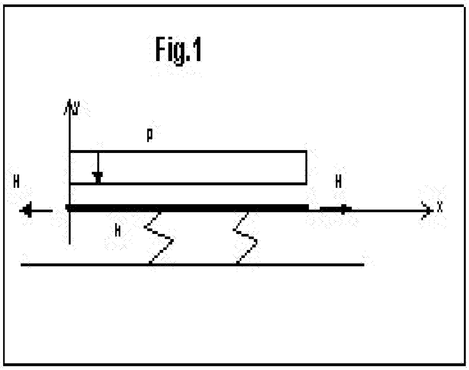
Fig.1
y
p
H
H
x
k

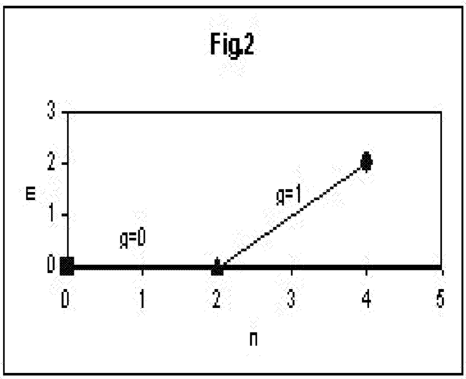
Fig.2
3
2
1
0
m
g=0
g=1
0
1
2
3
4
5
n

of all it is important to define the significance of the asymptotic approach from Cicala's point of view, in order to avoid confusion about the terms and the concepts. Afterwards the right procedure will be clarified using an application. The structural beam problem supplies a good example to show how it is possible to obtain the classical linear model and the nonlinear Von Karman approximation from a well defined group of equations. No aprioristic assumptions are introduced apart from the asymptotic one. Cicala's approach assigns a certain order of smallness to the variables and to the parameters involved in the problem and with respect to particular ratios of the physical dimensions of the structural element. Allowing for these definitions it is possible to classify the structural elements as indicated in [11],[12] and to define an approximate model peculiar for the single category. The order of smallness of the variables is worked out by the known magnitude of the coefficients of the considered equations. Consequently the starting equations are simplified neglecting the high order terms and the final approximate model is obtained. This procedure gives the possibility of introducing nonlinear terms of the same order as the principal ones in the model, without the need for a new execution. Every simplification is applied with respect to the order of the specific terms in the equation considered. In this case the initial relations are complete and the simplifications are introduced at the end of the procedure. In the subsequent sections, Cicala's approach will be applied to the derivation and justification of the linear (Eulero-Bernulli) and nonlinear (Von Karman) approximate beam models. Some interesting conclusion will be pointed out.

A different procedure([20],[21],[22],[23]) is summarized for the derivation of asymptotic approximate structural models. It starts by defining of the equilibrium equations in terms of displacements and the characteristic parameters of the elastic behavior of the material. Not dimensional coordinates and displacements are introduced in order to not dimensionalize the equations. A particular order of smallness is assigned to the located not dimensional parameters depending on the coherent and reasonable result obtained at the end of the procedure. Expanding the displacements in series of the small parameter and substituting them into the equilibrium equations, a series of problems with different degrees of approximation are worked out. Solving the P_0,P_1,$\cdots$ consecutive approximate problems, the displacements are calculated with an increasing level of approximation. An important paradox arises: starting from the linear system the approximate model concerns transversal displacements of the same order as the thickness, while starting from the non-linear system, under the same level of applied loads and the same boundaries, the approximate model involves transversal displacements of the same order of the thickness as the previous one. The cited authors observed that the linearized system is just a simplification. For this reason the conclusion could be incorrect. The same observation is also reported in a recent paper [24] together with any other justifications. A comparison with Cicala's procedure could be fruitful.

2.2 The asymptotic approach: Cicala's approach

The methodology based on the work of Prof. Cicala as reported in [11],[12],[13] for the unidimensional case and in [8],[9],[10] for the bidimensional one is described in the present section. The unidimensional configuration is considered here even though the approach is quite general and can be applied to different situations. The procedure can be summarized by the following main steps [5].

a) The definition of the initial equations: the strain-displacement equations and the local equilibrium equations are deduced from the tridimensional theory in its complete form according to [11],[14],[15],[16]. The basic assumption concerns the smallness of the strain sustained by the structural element, in order to remain in the elastic and reversible range. For this reason the constitutive stress-strain equations are limited to the classical linear ones. No conditions are imposed on the magnitude of the displacements and rotations that are the right unknowns of the problems. The strain-displacement equations are expressed in terms of the elongation and rotation of a principal line over which a local displacement is superimposed. This introduces some simplifications and separates the contribution of the global deformation from the contribution of the local deformation in the general expressions. The equations indicated here are considered as a whole and no substitution is made in order to try to express them in terms of the displacements.

b) The order of smallness of the variables involved is defined. All the variables are connected to the small parameter ϵ that gives the level of the maximum tension strain in the structural element. It is possible, as indicated in [12], to classify the structures on the basis of the order of that parameter.

c) Once the order of each coefficient in the equations has been established, it is possible to evaluate the order of smallness of the unknowns using a procedure reported in [1] and also in [17]. The order of smallness of the variable assumed as input data is introduced according to the classification of the structures and according to certain evaluations of the global behavior of the structure ([18]).

d) At this point each term in the equations considered has its specific order and they can be compared one another. The most important terms are collected and the others neglected in order to obtain an approximate expression for each equation. The evaluation of the important terms is extended also to the nonlinear ones. If nonlinear terms are of the same order as the principal ones they have to be considered in the approximate equation.

This procedure gives the possibility of taking into account all the terms necessary to build a consistent approximate model. Arbitrary simplifications are avoided because of the comparison of the order of the different terms. The system is more tractable and it doesn't oblige the operator to manage cumbersome equations. The approximate model is consistent with the conditions introduced before and it also gives an idea of the field of applicability of the approximation. Different approximate solutions are obtained changing the initial conditions. One more important concept, arising from Cicala's work, is the length of variation. In the application of the procedure , it is necessary to give an order of magnitude to the derivative operators as we did in the simple example of the previous section. Supposing that W' is the derivative of the variable W with respect to the coordinate x, and using L for the characteristic length of the structural element, it can be stated : $W' \approx W/L_\alpha$, $L_\alpha = L\epsilon^\alpha$. L_α is the length of variation of the variable W for that class of solutions ([10],[16]). The class of solution is characterized by the exponent α. In the subsequent derivation the value $\alpha = 0$ is assumed referring to the longitudinal coordinate. It concerns the case when the structural element is considered to be far from the constraints and from the points of loading application, and when variations of the section characteristics are smooth. All these situations have to be studied by means of classes of solutions different from than those described in this paper.

The procedure is applied to the definition of an approximate model of an isotropic initially straight beam considered with a solid or closed form cross section. A summary of the derivation is reported in [5].

For the beam-wise structures with massive section, the order of the ratio between the transverse dimension and the longitudinal dimension is $\epsilon^{0.5}$ ([12]). This also means that if the elastic characteristics are of the same order, the bending stiffness is of the same order as the torsional stiffness. Avoiding the fractional exponent, all the orders of magnitude are counted twice. For this reason in the case under consideration the classification parameter has an $h/L = \epsilon^1$ value. Assuming that the order of the variations of curvature of the structure is the same as the thickness (ϵ^1), the order of the resultant section moments is ϵ^5 with an order ϵ^2 for the bending stress σ.

If the resultant axial load has to be of the same ϵ^5 order as the other resultant quantities, the elongation of the reference line, e_0 , is forced to assume the order ϵ^3. In this case the tensile stress is of higher order with respect to the bending one and can be neglected in the derivation of the model. The resultant behavior is typically not-extensional (see [5]).

A different classification is obtained if the torsional load is of higher order than before because of the combination of the transversal load and of the deflection. Regarding the first situation described in [5], the torsion problem is reduced to the St.Venant approximation, the longitudinal strain is reduced to the bending terms, and the $\sigma_{11}, \sigma_{22}, \sigma_{12}$ stresses can be neglected because of an order higher than the others. The shear effect is neglected because the corresponding stresses are of an order higher than the bending stresses. The stresses σ_{01}, σ_{02} are reduced to the torsional and warping terms. In this way the section is supposed to be undeformable in its plane. If the resultant loads over the section are considered, the equations are reduced to:

$$\begin{aligned} N_{0,0} + p_0 = 0, \quad N_{1,0} + p_1 = 0, \quad N_{2,0} + p_2 = 0, \\ M_{1,0} + q_1 = N_2, \quad M_{2,0} + q_2 = -N_1, \quad M_{0,0} + q_0 = 0, \end{aligned} \tag{13}$$

with the resultant loads and moments such that: $N_\alpha \equiv \epsilon^5; M_\alpha \equiv \epsilon^5$. The approximation consists of the Eulero-Bernoulli case.

With the same classification of the variables, the elongation e_0 can reach the order ϵ^2. The equations are modified as follows:

$$2\epsilon_{00} = 2(e_0 + \Delta\omega_1\xi_2 - \Delta\omega_2\xi_1), 2\epsilon_{01} = u_{0,1} - \Delta\omega_0\xi_2, 2\epsilon_{02} = u_{0,2} + \Delta\omega_0\xi_1. \tag{14}$$

The resultant loads and moments become: $N_0 \equiv \epsilon^4; N_{1,2} \equiv \epsilon^5; M_\alpha \equiv \epsilon^5$. The equilibrium equations are reduced to:

$$\begin{aligned} N_{0,0} + p_0 = 0, \quad N_{1,0} + N_0\omega_2 + p_1 = 0, \quad N_{2,0} - N_0\omega_1 + p_2 = 0, \\ M_{1,0} + q_1 = N_2, \quad M_{2,0} + q_2 = -N_1, \quad M_{0,0} + q_0 = 0, \end{aligned} \tag{15}$$

with the bending moments of the same order as the torsional moment. The relations are now nonlinear due to the presence of a nonlinearity of the Von Karman type. It is necessary to make some remarks regarding the order of the variables involved. Expressing the elongation and the curvatures of the reference line with respect to the three displacements u, v, w and a rotation ϕ in the initial triad, the two already mentioned conditions give the classical linear beam model and the classical nonlinear beam model. In fact as reported in [17], and in [5], the input variables are:

$$\begin{aligned} e_0 &= u_{,s} + 1/2(v_{,s}^2 + w_{,s}^2) + \cdots, \\ \omega_0 &= \phi_{,s} + \cdots, \\ \omega_1 &= -w_{,ss} + \cdots, \\ \omega_2 &= v_{,ss} + \cdots. \end{aligned} \tag{16}$$

If the transversal displacements have an order of magnitude higher than the section dimensions, that is : $u = \epsilon^3; v, w = \epsilon^2; \phi = \epsilon^2$, the result is : $e_0 = \epsilon^3; \omega_\alpha = \epsilon^2$, and we get the same relations as in the linear model. It is important to note that the in plane displacement is of one order higher than the transverse one. If the transverse displacements are of the same order as the transverse dimension of the beam, that is: $u = \epsilon^2; v, w = \epsilon^1; \phi = \epsilon^1$, then: $e_0 = \epsilon^2, \omega_\alpha = \epsilon^1$. The approximation corresponds to a nonlinear model.

The second situation described in [5] is concerned with a different approximation. The main difference with the previous one consists in the introduction of the shear effect. The shear stresses are as important as the torsion stresses which are no more distributed according to the assumptions of S.Venant. The beam model is in the Timoshenko complete form and the last resultant equilibrium equation contains the effect of bending moments.

2.3 Different approach

A brief description of the procedure ([20]) is reported. The reader is addressed to the cited references for details. This method can be summarized through the following steps:

a) Strain-displacement equations, constitutive equations and equilibrium equations are defined as a choice of the operator in the linear or the nonlinear sense (cited references). The equilibrium equations are expressed in terms of the displacements by substitution of the strain-displacement and constitutive equations. The resulting equations are considered as starting points in the procedure.

b) The relevant variables are introduced in order to not dimensionalize the quantities involved in the problem. The position coordinates of each point of the element are not dimensionalized with respect to the corresponding geometric dimension of the element itself. An order of smallness is given to some of these ratios with respect to the shape ratio. It is defined as the ratio between the transverse and longitudinal dimension of the element.

c) The starting equations are not dimensionalized by introducing the ratios defined previously. At this stage certain peculiar not dimensional parameters are identified.

d) The displacements are expressed in series of a small parameter as: $V = V^0 + \epsilon V^1 + \epsilon^2 V^2 + \ldots$ and put into the equations defined in the previous step. Collecting the terms of the same order, a sequence of approximate problems as $P_0, P_1, \cdots$ is defined and their solution gives rise to different approximate structural models.

The expressions of the strain components limited to the linear approximation are:

$$\epsilon_{ii} = u_{i,i}, \qquad \epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{17}$$

with $i, j = 1, 2, 3$ and without any summing convention. The u_α are the components of displacement in a Cartesian system where the x_α are the coordinates. The linear expressions are used only for convenience in order to clarify the procedure as in [20] and to derive the classical linear model. The constitutive relations in an isotropic and elastic case are:

$$\begin{aligned} \sigma_{11} &= (\lambda + 2\mu)\epsilon_{11} + \lambda\epsilon_{22} + \lambda\epsilon_{33}, & \sigma_{12} &= 2\mu\epsilon_{12}, \\ \sigma_{22} &= \lambda\epsilon_{11} + (\lambda + 2\mu)\epsilon_{22} + \lambda\epsilon_{33}, & \sigma_{13} &= 2\mu\epsilon_{13}, \\ \sigma_{33} &= \lambda\epsilon_{11} + \lambda\epsilon_{22} + (\lambda + 2\mu)\epsilon_{33}, & \sigma_{23} &= 2\mu\epsilon_{23}, \end{aligned} \tag{18}$$

where λ, μ are the Lamé parameters. The linear equilibrium equations are:

$$\sigma_{11,1}+\sigma_{21,2}+\sigma_{31,3}+f_1 = \sigma_{12,1}+\sigma_{22,2}+\sigma_{32,3}+f_2 = \sigma_{13,1}+\sigma_{23,2}+\sigma_{33,3}+f_3 = 0. \tag{19}$$

Substituting the strain-displacement relations in the constitutive relations and then in the equilibrium equations, the expanded form of the starting equations can be obtained .

Restricting,as an example, the structural behavior to the $1-3$ plane with u_1, u_3 and f_3 active and without variation along the x_2 coordinate, the problem is reduced to :

$$\begin{aligned} \mu(u_{1,11} + u_{1,33}) + \mu(u_{1,11} + u_{3,13}) + \lambda(u_{1,11} + u_{3,31}) = 0, \\ \mu(u_{3,11} + u_{3,33}) + \mu(u_{1,31} + u_{3,33}) + \lambda(u_{1,13} + u_{3,33}) + f_3 = 0. \end{aligned} \tag{20}$$

The referring value of the displacements is indicated with W for u_3, V for u_2 and with U for u_1, while L is the reference length and h the reference thickness.The subsequent not dimensional variables are defined as: $\bar{u}_3 = u_3/W, \bar{u}_2 = u_2/V, \bar{u}_1 = u_1/U, \bar{x}_3 = x_3/h, \bar{x}_1, \bar{x}_2 = x_1, x_2/L, \bar{f}_3 = f_3/F_3$,

$$\begin{aligned} \epsilon^2 \left[\bar{u}_{1,11} + (1+\beta)\bar{u}_{1,11}\right] + \epsilon\gamma(1+\beta)\bar{u}_{3,31} + \bar{u}_{1,33} &= 0, \quad (21) \\ \epsilon^2 \left[(1+\beta)\bar{u}_{1,13} + \epsilon\gamma\bar{u}_{3,11}\right] + (2+\beta)\epsilon\gamma\bar{u}_{3,33} + \epsilon\gamma\alpha\zeta\bar{f}_3 &= 0. \end{aligned}$$

The not dimensional equations are derived emphasizing some not dimensional parameters, such as: $\beta = \lambda/\mu$, $\gamma = W/U$, $\epsilon =$

h/L, $\alpha = hF_3/\mu$, $\zeta = h/W$, as reported in [21]. At this point no hypotheses are introduced about the order of smallness of the not dimensional parameters. Considering such as in [20] that $U = \epsilon W$, $W = h$ (this condition is equivalent to introducing the transverse displacement of the same order as the thickness and the in plane displacement of an order higher than the transverse displacement) and $\alpha = \epsilon^4$, the not dimensional equations are in a form suitable for the asymptotic derivation. The approximate problems of different level, can be worked out:

$$\begin{aligned} \epsilon^2 \left[\bar{u}_{1,11} + (1+\beta)\bar{u}_{1,11}\right] + (1+\beta)\bar{u}_{3,31} + \bar{u}_{1,33} &= 0, \\ \epsilon^2 \left[(1+\beta)\bar{u}_{1,13} + \bar{u}_{3,11}\right] + (2+\beta)\bar{u}_{3,33} + \epsilon^4 \bar{f}_3 &= 0. \end{aligned} \tag{22}$$

Assuming, as to the displacements, that the expansion is as follows:

$$\bar{u}_k = u_k^0 + \epsilon u_k^1 + \epsilon^2 u_k^2 + \ldots, \tag{23}$$

with $k = 1, 3$, and substituting them into the not dimensional equations, we get

- P^0 level:

$$(1+\beta)u^0_{3,31} + u^0_{1,33} = 0 \quad u^0_{3,33} = 0; \tag{24}$$

- P^1 level:

$$(1+\beta)u^1_{3,31} + u^1_{1,33} = 0 \quad u^1_{3,33} = 0; \tag{25}$$

- P^2 level:

$$\begin{aligned} (1+\beta)u^0_{1,11} + u^0_{1,11} + (1+\beta)u^2_{3,31} + u^2_{1,33} &= 0, \\ (1+\beta)u^0_{1,13} + u^0_{3,11} + (2+\beta)u^2_{3,33} &= 0, \end{aligned} \tag{26}$$

and so on. The solution to the problem P^0 is a displacement of Kirchoff-Love with a component solution of a bending problem and a component solution of a membrane one (see [20] for details of complete form problem).

Solving the consecutive approximate systems on the basis of the corresponding boundary conditions not reported here, the behavior of the displacements is found to be the same as in the linear expected case. In the cited reference the same assumptions are used in conjuction with the nonlinear tridimensional equations in order to introduce the Von Karman nonlinearity in the model. This apparent paradox is justified in [23] where the classical linear model is valid if the transverse displacement is of one order higher than the thickness and the transverse load is higher by one order than the one considered at the beginning. The introduction of the correct hypotheses and the right terms in the approximate models is made possible by a process of rearranging the procedure using each time different starting equations and comparing the different results. The starting equations are introduced as in the intention of the operator and the results are correct if and only if all the necessary terms are taken into account from the beginning.

A discussion of the results is necessary in order to clarify the indicated paradox. It is obvious that when using a linearized system deduced from a non-linear one, arbitrary simplifications are introduced that influence the final solution. This cannot be an effective explanation for the paradox. In fact, the first approximation of a linearized system does not usually coincides with the first approximation of a non-linear system, and the approximate linearized solution is not representative enough of an approximate solution for the non-linear case. This is true only in specific cases and under specific conditions (as can be verified using a very simple algebraic system). The field of applicability in the linearized situation cannot be considered as valid because of a priori neglecting of some terms. Its validity is derived by the linearization assumptions. For these reasons the paradox is not effective.

3 Cicala's approach and the asymptotic development convenience

The general perturbative expansions in series of a small parameter are useful to obtain some results both qualitative and quantitative, even if they could be divergent ([25], [27]). It is not possible to know a priori whether such a kind of expansion is convergent or not. This can be done only after computing the subsequent term. Considering now a very simple example ([25]) such as :

$$f = 1 + \epsilon f^3, \tag{27}$$

the solution is in the form:

$$f(x, \epsilon) = f_0(x) + \epsilon f_1(x) + \epsilon^2 f_2(x) + \ldots, \tag{28}$$

where the f_α are linearly independent functions obtained through the usual procedure as:

$$f_0 = 1, f_1 = f_0^3 = 1, f_2 = 3f_0^2 f_1 = 3, f_3 = 3f_0^2 f_2 + 3f_0 f_1^2 = 12. \tag{29}$$

The approximate solution is: $f = 1 + \epsilon + 3\epsilon^2 + 12\epsilon^3 + \ldots$.

A comparison with Cicala's approach ([10],[4]) can be done. The first step is the determination of the "fundamental" solution to the problem. It derives from the classification of the unknowns that identifies the most important terms in each equation of a general system $L(g) = 0$. It is reduced to a subsystem $L_{11}(g_1) = 0$ (the fundamental one) where the g_1 are the principal unknowns in the n_{g1} principal equations. The general system is subdivided into a specific form (that Cicala indicated as "scalarized form") such as:

$$L(g) = L_{11}(g) + \ldots + L_{12}(g) + \ldots + L_{13}(g) + \ldots = 0, \tag{30}$$

if the number of equations is the same in the various groups, or such as:

$$\begin{gathered} L_{11}(g_1) + \ldots + L_{12}(g_1, g_2) + \ldots + L_{13}(g_1, g_2, ..) + \ldots = 0, \\ L_{21}(g_2) + \ldots + L_{31}(g_2, g_3) + \ldots = 0, \\ \ldots, \end{gathered} \tag{31}$$

where at any step the number of unknowns and consequently of equations are modified. The different groups contain terms of the same order and of one order higher than the previous. The scalarization is a logic consequence of the classification of the unknowns. From the system L_{11} the fundamental solution $\bar{g}$ is obtained. A successive approximation is calculated considering the terms of one order higher than the fundamental ones that is considering the group L_{12} (referring to the first kind) in the solution. The values of the $\bar{g}$ unknowns obtained in the previous step are assigned to the unknowns contained in that group as showed by:

$$L_{11}(g') + L_{12}(\bar{g}) = 0, \tag{32}$$

where g' is the new approximation. Another approximation g'' is obtained as follows:

$$L_{11}(g'') + L_{12}(g') + L_{13}(\bar{g}) = 0 \tag{33}$$

and so on. The successive approximations can be obtained from a system written in the form:

$$L_{11}(g') + L_{1\alpha}(\bar{g}) = 0, \quad L_{11}(g'') + L_{1\alpha}(g') = 0, \tag{34}$$

without any loss of information. $L_{1\alpha}$ contains all the successive terms. With reference to the initial simple example, Cicala's procedure leads to $f_0 = 1$ as the fundamental solution to the equation with the unknown of order 0. The scalarization produces the relation: $L_{12} = \epsilon f^3$. The successive approximations are :

$$\begin{aligned} f_0 &= 1, \\ f' &= 1 + \epsilon f_0^3 = 1 + \epsilon, \\ f'' &= 1 + \epsilon (f')^3 = 1 + \epsilon + 3\epsilon^2 + \cdots, \\ f''' &= 1 + \epsilon (f'')^3 = 1 + \epsilon + 3\epsilon^2 + 12\epsilon^3 + \cdots. \end{aligned} \tag{35}$$

The solution corresponds to a power series in ϵ as the previous one and as observed by Cicala [10]. It is interesting to observe that the successive approximations are obtained using a point map ([6]) such as:

$$u_{k+1} = F + P(u_k, \epsilon), \quad P(u_k, \epsilon) = \epsilon u_k^3, \tag{36}$$

with F representing the fundamental solution. It is advisable to indicate whether the computation of a further approximation is convenient or not. Assuming $\bar{u}$ as the exact solution such that: $\bar{u} = F + P(\bar{u}, \epsilon)$, the successive approximation are :

$$\bar{u} = u_0 + l_0 = u_1 + l_1 = u_2 + l_2 = \ldots = u_N + l_N \qquad (37)$$

and:

$$u_0 = F, u_1 = F + P(u_0, \epsilon), u_2 = F + P(u_1, \epsilon), \ldots, u_N = F + P(u_{N-1}, \epsilon). \qquad (38)$$

Substituting then into the relations we get:

$$\begin{aligned} u_1 + l_1 &= F + P(u_0 + l_0, \epsilon) = F + P(u_0, \epsilon) + \partial P(u_0, \epsilon) l_0, \\ u_2 + l_2 &= F + P(u_1 + l_1, \epsilon) = F + P(u_1, \epsilon) + \partial P(u_1, \epsilon) l_1, \\ \ldots &\quad , \\ u_N + l_N &= F + P(u_{N-1} + l_{N-1}, \epsilon) = F + P(u_{N-1}, \epsilon) + \partial P(u_{N-1}, \epsilon) l_{N-1}, \end{aligned} \qquad (39)$$

and therefore we obtain a series of successive errors:

$$\begin{aligned} l_1 &= \partial P(u_0, \epsilon) l_0, \qquad (40) \\ l_2 &= \partial P(u_1, \epsilon) l_1 = \partial P(u_1, \epsilon) \partial P(u_0, \epsilon) l_0, \\ l_3 &= \partial P(u_2, \epsilon) l_2 = \partial P(u_2, \epsilon) \partial P(u_1, \epsilon) \partial P(u_0, \epsilon) l_0, \\ \ldots &\quad , \\ l_N &= \partial P(u_{N-1}, \epsilon) l_{N-1} = \partial P(u_{N-1}, \epsilon) \partial P(u_{N-2}, \epsilon) \ldots \\ &\ldots \partial P(u_1, \epsilon) \partial P(u_0, \epsilon) l_0 = K_N l_0. \end{aligned}$$

The error tends to reduce if and only if the absolute value of K_N is less than the unity: $\mid K_N \mid < 1$. In this case the calculation of successive approximations is convenient, otherwise the error increase and the solution diverges. If this is the case the solution is limited to the fundamental. An extension of the general multidimensional situation is possible. The symbol ∂ assumes the meaning of Jacobian of the transformation. The error is diminishing if all the eigenvalues of the Jacobian matrix are in modulus less than the unity ([26], [19]). In this case the calculation of the successive approximation is convenient. The Liapunov exponent is immediately calculated:

$$\Lambda_i = \lim_{N\to\infty} \frac{1}{N} \ln \mid \lambda_i^N \mid, \tag{41}$$

with λ_i representing the i-th eigenvalue of the Jacobian at the N step as indicated. If the condition: $\Lambda_i < 0$ holds for all the Λ_i, the solution is convergent. If only one of the exponents is higher than zero the solution is divergent.

A simple linear bidimensional case follows. The system is:

$$(1+\delta^2)x + 2\delta y = 1, \qquad \delta x + (3+\delta)y = -\delta, \tag{42}$$

it is a linear system with coefficient function of the small parameter. The exact solution is:

$$x = \frac{3+\delta+2\delta^2}{3+\delta+\delta^2+\delta^3}, \quad y = \frac{-\delta(2+\delta^2)}{3+\delta+\delta^2+\delta^3}. \tag{43}$$

Allowing for Cicala's definition its fundamental solution turns out to be:

$$x = 1, \quad \delta x + 3y = -\delta, \tag{44}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 \\ -\delta/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ -\delta \end{pmatrix} = \begin{pmatrix} 1 \\ -2\delta/3 \end{pmatrix}. \tag{45}$$

The successive approximation is obtained from the scalarization as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 \\ -2\delta/3 \end{pmatrix} + \begin{pmatrix} -\delta^2 & -2\delta \\ \delta^3/3 & 2\delta^2/3 - \delta/3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{46}$$

In this case the Jacobian is independent of the actual value of the unknowns (since the system is linear). Table 3 and 4 can be arranged ($\lambda^{(1)}$ indicates the eigenvalue at the first step).

Tab.3

δ	$\lambda^{(1)}$	*exact*	x^0	x'	x''	x'''
-0.02	-0.0004	1.0001	1	1.0001		
	0.0069	0.0134	0.0133	0.0134		
-0.05	-0.0023	1.0009	1	1.0008	1.0009	
	0.0181	0.0339	.0333	0.0339	0.0339	
-0.1	-0.0086	1.0038	1	1.0033	1.0038	
	0.0386	0.0691	0.0667	0.0690	0.0691	

Tab.4

δ	$\lambda^{(1)}$	*exact*	x^0	x'	x''	x'''
0.02	-0.0004	1.0001	1	1.0001		
	-0.0064	-0.0132	-0.0133	-0.0132		
0.05	-0.0028	1.0008	1	1.0008		
	-0.0147	-0.0328	-0.0333	-0.0328		
0.1	-0.0167	1.0029	1	1.0033	1.0029	
	-0.02	-0.0646	-0.0667	-0.0646	-0.0646	

Different values are given to the parameter δ in order to show the behavior of the system, but it is important to notice that the asymptotic developments are significative only for small values of the parameter itself. A tentative criterium can be introduced: **The Asymptotic Development Convenience**. *$F(x, \delta) = 0$ is a linear system depending on a small parameter δ with n equations in n unknowns. The x_0 is the fundamental solution in the sense of Cicala and it is obtained from the subsystem $F_{11}(x_0, \delta) = 0$ after the classification that transforms the system into the scalarized form as:*

$$F(x, \delta) = F_{11}(x, \delta) + F_{12}(x, \delta) + F_{13}(x, \delta) + \cdots, \tag{47}$$

assuming that the $F_{11}(x_0, \delta) = 0$ contains all the unknowns of the initial one F. The successive approximations are calculated from *the sequence:*

$$F_{11}(x_{k+1},\delta) = -F_{1\alpha}(x_k,\delta) \Rightarrow x_{k+1} = -F_{11}^{-1}F_{1\alpha}(x_k,\delta) = \bar{F}(x_k,\delta), \tag{48}$$

that is convergent to the exact solution only if, at each step, the eigenvalues of the Jacobian of $\bar{F}$ are in modulus less than unity.

If this is the case the asymptotic procedure originates a better approximation at each step. Exceptional cases originated under specific conditions are out of the scope of this preliminary analysis which is limited to the well defined ones. Other conditions are still under investigation and are not dealt with in this paper.

4 Concluding remarks

Cicala's asymptotic approach is presented and described with the help of an application to the beam structural problem. The beam approximate models are derived and the approximations are justified. The general procedure is compared to a different one and the obtained results are comparable if and only if the initial assumptions are the same. It is important to note that the initial hypotheses have to be the same not only with respect to the order of the displacements and other variables but also with respect to the same starting equations. While the first method allows to take into consideration all the terms of the same order (linear and nonlinear) included in the complete initial equations by checking their order of smallness, the different cited procedure can start with different initial relations not always representative of the phenomena. After a comparison between the different final results, this latter procedure can realize the discrepancies and modify the starting equations accordingly, but only at the end of the procedure. In Cicala's approach the operator has to wonder whether the terms considered in an approximate model are complete or whether there are some other terms important for that problem. All the fundamental terms, linear and nonlinear, are collected consistently with the initial assumptions: none of them is forgotten. The concept of the length of variation is also an other very important aspect in Cicala's derivation. It enables the operator to deal with different classes of solutions with regards to different variation characteristics of the problem. A new connection with general mathematical theory is found thanks to Cicala's derivation. Quite a new criterium is defined proving the convenience of successive asymptotic approximations.

Contributions

The general structure of the paper was arranged by the two authors together in order to give some ideas about the work of Prof. Cicala. The different aspects were reconsidered with the aid of specific original applications and the key points were emphasized as summarized in the conclusions. The first author was mainly involved into the definition of the first part with the general presentation of the methodology and the indication of the specific point of view considered. The second author was mainly involved into the development of the second part and into the original formulation presented in the third part, that is also a part of his PhD activity.

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