General approach to non-linear problems in elasticity

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presentata nell'adunanza 17 gennaio 2007

Abstract. *The problems of elasticity, where non-linearity derives from geometrical effects of combination between deformations and stresses, are studied with a tensor three-dimensional approach. Deformation effects are analysed by means of the Cristoffel symbols in the deformed medium with respect to that of the elastic medium, in a Lagrangean approach to the structural behaviour, where displacements are chosen as unknowns of the problem. The non-linearity in the equilibrium equations is completely taken into account in a proposed equation system. The interaction between deformations and stresses is particularly important in those structures where are present components with small thickness and wide extensions.*

Keywords: elastic instability, structural behaviour, Cristoffel symbols, tensor calculus, equilibrium equations, non-linear problem.

Riassunto. *Il problema della elasticità, dove le non linearità derivano da effetti geometrici di combinazione fra deformazioni e sforzi, vengono studiati attraverso un approccio tensoriale tridimensionale. Gli effetti delle deformazioni vengono studiati per mezzo dei simboli di Cristoffel nel mezzo deformato in rapporto al mezzo elastico, in un approccio Lagrangiano al comportamento strutturale, nel quale gli spostamenti sono assunti come incognite del problema. Le non linearità nelle equazioni di equilibrio sono prese in considerazione completamente in un sistema di equazioni proposto. L'interazione fra deformazioni e sforzi è particolarmente importante nelle strutture in cui sono presenti componenti con piccoli spessori e ampie estensioni.*

Parole chiave: instabilità elastica, comportamento strutturale, simboli di Cristoffel, calcolo tensoriale, equazioni di equilibrio, problema non lineare.

1. General remarks on structural approaches

The elasticity quoted in the title means small material strains. This doesn't mean small displacements in a structure. The aim of this paper is therefore to discuss the effects of the non-linearity due to the combination between deformations (as bends and twists) and stresses, when strains are small.

The A., [18], introducing from a general point of view the structural problem, reviewed considerations that are here reported in a part very useful also to our purposes. Structural problems, with the exception of those concerning the determination of fatigue effects, where a discrete nature of materials is involved, can be faced with mathematical models, that assume materials as continua. Disregarding thermal, piezoelectric and electromagnetic loads, here not considered, they can be divided into two families:

a) the determination of the structural behaviour under mechanical external loads, which, generally speaking, has a dynamic nature, whose time history is known, and also under external constraints, of a known time history; by structural behaviour we mean both the evolution of a structure, studied by a structural analysis, and the stability of such an evolution;

b) as an increasing parameter is considered, the determination of critical stability limits of self-induced deformations, generally speaking of a dynamic nature, where external loads are generated because of such deformations themselves, in the presence of external force sources, must be determined. We are here only concerned with point a). Structural analysis, from the most general point of view, should be considered a three–dimensional problem, where non Cartesian co–ordinate systems can also be adopted. As far as point a) is concerned, some precise statements should be made. External loads, in a three-dimensional structure, can be applied as distributions of a specific surface force (pressure) on portions of the boundary or as distributions of a specific volume force (for instance weight or inertia force) on portions of the definition field. Both can be put under the form of distributions of specific volume force by means of "generalised functions", [4] and [12]. External constraints can assume the nature of impressed deformations applied to regions of the field of definition– local unitary deformations having a non elastic nature such as dislocations, plastic deformations, thermal dilatations – or of imposed positions, to points of the field or of the boundary, coherent with the desired behaviour of the structure as far as continuity is concerned. All these constraints can be put under the form of conditions on regions of the field of definition, by means of "generalised functions", [4], and [12]. When the material is considered a continuum, the structural problem has at least three unknown functions, the displacement components, that depend on three space coordinates and, in dynamics, on the time. In the case of thin structures – a dimension smaller than the others – or long structures – two dimensions smaller than the other –, by means of axioms concerning stress or stress behaviour, as suggested by some great and brilliant scientists as Bernoulli, de St. Venant, Timoshenko and so on, it is possible to reduce the number of space co-ordinates to two or one, respectively, in approximate theories where the equilibrium equations concern volumes that are infinitesimal in two or one dimensions, respectively – the minimum number of the unknown functions depends on the degrees of freedom permitted by the axioms. The resolving equations in such unknown functions in a rigorous approach result not linear. Non-linearity in a mathematical model can derive from three sources – i) changes in the form of the body due to displacements, ii) geometrical relations between displacements and strains and iii) non linear behaviour of the materials, concerning in particular relations between strains and stresses. In order to present a sample of i) and ii) we can refer to the analysis of curved shells due to Cicala, [5], [6] and [11], where in a two dimensional analysis are given the following equilibrium equations:

$$\frac{1}{h_a}\frac{\partial}{\partial\xi_a}N_a+\frac{1}{h_b}\frac{\partial}{\partial\xi_b}N_{ba}+\rho_{ga}\left(N_{ab}+N_{ba}\right)+\rho_{gb}\left(N_a-N_b\right)+\rho_a Q_a+\rho_{ab}Q_b+p_a=0$$
$$\frac{1}{h_b}\frac{\partial}{\partial\xi_b}N_b+\frac{1}{h_a}\frac{\partial}{\partial\xi_a}N_{ab}+\rho_{gb}\left(N_{ab}+N_{ba}\right)+\rho_{ga}\left(N_b-N_a\right)+\rho_b Q_b+\rho_{ab}Q_a+p_b=0$$
$$\frac{1}{h_a}\frac{\partial}{\partial\xi_a}Q_a+\frac{1}{h_b}\frac{\partial}{\partial\xi_b}Q_b+\rho_{gb}Q_a+\rho_{ga}Q_b-\rho_a N_a-\rho_b N_b-\rho_{ab}\left(N_{ab}+N_{ba}\right)+p=0$$
$$\frac{1}{h_a}\frac{\partial}{\partial\xi_a}M_a+\frac{1}{h_b}\frac{\partial}{\partial\xi_b}M_{ba}+\rho_{ga}\left(M_{ab}+M_{ba}\right)+\rho_{gb}\left(M_a-M_b\right)=Q_a$$
$$\frac{1}{h_b}\frac{\partial}{\partial\xi_b}M_b+\frac{1}{h_a}\frac{\partial}{\partial\xi_a}M_{ab}+\rho_{gb}\left(M_{ab}+M_{ba}\right)+\rho_{ga}\left(M_b-M_a\right)=Q_b$$
$$\rho_{ab}\left(M_b-M_a\right)+\rho_a M_{ab}-\rho_b M_{ba}+N_{ab}-N_{ba}=0$$

In such equations ξ_a, ξ_b are orthogonal curvilinear co-ordinates on the medium surface, h_a, h_b are the modula of the derivatives of the position respect to the co-ordinates, $N_a, N_b, N_{ab}, N_{ba} - Q_a, Q_b - M_a, M_b; M_{ab}, M_{ba}$ are respectively normal, transverse and bending and twisting stresses on an element of shell, $\rho_a, \rho_b, \rho_{ab} - \rho_{ga}, \rho_{gb}$ are respectively curvatures and geodetic curvatures of the medium surface. When curvatures and geodetic curvatures are composed by two contributions, one due to the form of the initial reference system and one due to the deformations of the body, this last part must be expressed as stresses in terms of the unknown displacements. So the relative terms of the equilibrium equations result non linear. Since we are concerned only with elasticity, we can reduce our analysis to the field of small strains and in consideration of the most frequent behaviour of materials to linear stress-strain relations. The dynamic nature of the general structural problem requires the expressions of both the internal forces and the inertial forces. The inertial forces require the evaluation of the accelerations due to the displacements as functions of time. The non-linearity of the effects of the displacements make necessary an analysis based or on an Euler's or an Lagrange's approach (see [8], Cap. IX, § 66). Because of the possibility of deriving a dynamic analysis starting from a static one by means of the d'Alembert principle, we shall usually refer to static analyses, quoting the dynamic aspects, if necessary. The static problem of the interaction between deformations and internal stresses was already faced by the A., [7], with reference to the linear approaches, in an introduction that is very useful also to our purposes and is here reported. The static problem of a structure submitted to a given set of loads F (whose intensity is represented by the value P of one of them) consists after all in determining the deformed configuration of the same structure and the value of P for which the failure occurs. Such a problem is solved through differential equations based on a proper schema of the given structure, where the phenomena to be analytically processed are pointed out. To the selected schema is associated a system of co-ordinates established in the space where the structure is defined. Assuming as a main unknown the expression related to the displacements of the structure (for example the shift of the axis for the mono-dimensional scheme and the mean surface for the bi-dimensional one), the problem will usually lead to a system of differential equations, equal in number to the unknown functions, with proper conditions at the neighbourhood. Such equations in the displacements are reached through auxiliary unknown (strain, stresses, etc.) and expression (equilibrium conditions of an infinitesimal element, relation between strain and stresses of the material and deformation state at each point of the structure in function of the unknown functions). When: i) linear relations exist between strain and stresses, ii) displacements are such that in the evaluation of the deformation state, the non-linear terms of the displacements themselves are negligible, iii) the load application is gradual and the displacement accelerations are small, so as to avoid inertial reactions to the displacements, under the conditions i), ii) and iii), for a given value of P, the expressions of the body deformation state in function of the unknown are linear.

The equations of equilibrium, instead, consist of two types of terms: the first is formed by derivatives, with respect to the co-ordinates, of stresses (resultants of tensions) and gives account of the increments of such quantities through the infinitesimal element; the other is formed by products between internal forces (or resultants of tensions) and local bends or twists of the body and gives account of the stress components unbalanced because of the angle between face and face of the infinitesimal element under consideration. The stresses (or resultants of tension) and the angles, to be introduced into the equations of equilibrium, are those one which occur at the equilibrium itself. The angles then will consist of two terms: Δ_i, corresponding to the initial configuration, and $\overline{\varepsilon}$, due to the elastic deformation. Symbolically we can write, [7]:

$$\Delta = \Delta_i + \overline{\varepsilon}$$

The local stress values, expressed in terms of strains, can be symbolically represented by $\overline{\varepsilon}$.

The products between angles and stresses then can be represented in the form:

$$\overline{\varepsilon}\left(\Delta_i + \overline{\varepsilon}\right) = \Delta_i\overline{\varepsilon} + \overline{\varepsilon}^2$$

or rather they contain linear terms in $\overline{\varepsilon}$ and square terms. The equation thus obtained can be reduced to a linear form, starting from a known reference condition of equilibrium, symbolically represented by ε_0 and generated by a distribution of loads $F-\psi$, whose intensity is still represented by P and such that in it stresses or displacements are proportional to P.

The problem consists then in determining the effect, starting from the reference conditions, of the application of loads ψ

Symbolically, [7]:

$$\overline{\varepsilon} = P\frac{d\varepsilon_0}{dP} + \varepsilon$$

On the other hand, the angles can be symbolically expressed through:

$$\Delta = \Delta_i + P\frac{d\varepsilon_0}{dP} + \varepsilon$$

and the products give place to the following terms, [7]:

$$\left(P\frac{d\varepsilon_0}{dP} + \varepsilon\right)\left(\Delta_i + P\frac{d\varepsilon_0}{dP} + \varepsilon\right) = \Delta_i P\frac{d\varepsilon_0}{dP} + P^2\left(\frac{d\varepsilon_0}{dP}\right)^2 + \Delta_i\varepsilon + 2P\frac{d\varepsilon_0}{dP}\varepsilon + \varepsilon^2$$

If now we neglect the terms in ε^2 , it follows that the first two terms are constant, the third is linear and the fourth is linear in ε , but not in P and ε .

The linear equations in the displacements are therefore of the following matrix form, [7]:

$$L^*\left(w\right) + PL\left(w\right) = f \, , \tag{1.1}$$

related to the element of unitary dimensions. In the (1.1) w is a matrix (column) of unknown function, in number necessary and sufficient to define, in the scheme used, the deformed configuration of the structure; $\mathscr{L}$ is a linear operator, represented by a (square) matrix of linear differential operators generally with variable coefficients, acting on the w and such that $\mathscr{L}(w)$ represents the differentials of the elastic stress resultants by deriving the w; L is a linear operator of $\mathscr{L}$-type, but of lower order, and such that PL(w) represents the forces generated by the combination of the internal stress resultants and deformations, except for the non-linear terms; P represents the intensity of the applied loads assumed to be proportional to it; f is a matrix (column) of known functions. At the basis of the (1.1) there is a system of co-ordinates in number necessary and sufficient to locate biunivocally, in a proper schema, every "point" of the structure.

As to the validity of all the later considerations it is necessary that in any equation synthesized by the matrix equation (6.1), the various terms express forces acting on the element of unitary dimensions. To the equation (1.1) are also associated boundary conditions which translate the respective constraint conditions are linear, because placed either on displacements or on stress resultants (which are stated above are linear in the displacements):

$$C_i\left(w\right) = g_i \tag{1.2}$$

(defined on the boundary of S of V) where the g_i are matrices of known functions. The better approach to comprehend in a rigorous way the problem of the combinations between deformations and stresses is the tensor calculus. We shall make reference only to the tensor calculus applied to three dimensional problem, leaving to the reader the extension, if desired, to one or two-dimensional approaches. The minimum number of unknowns is three – the displacements – that can be determined by means of the three equilibrium equations. Such equations contain six stress components that can be expressed by means of six equations as functions of six deformation components (stress–strain relations). The deformation components can be expressed by means of six equations as functions of the displacement components (deformation–displacement relations). The problem can be considered as containing fifteen unknowns with fifteen equations. The equilibrium to be considered concerns the configuration under deformations, where the most useful reference system is curvilinear. The tensor calculus, and in particular the concept of covariant derivative, that utilises the Cristoffel symbols, results particularly indicated.

2. Approach by means of the tensor calculus

2.1. General concepts

The tensor calculus, or absolute differential calculus, begins with the Gauss's researches on curved surfaces. The tensor calculus of Euclidean spaces, applied in elasticity and in the theory of special relativity, is to be distinguished from the tensor calculus of curved or Riemannian spaces, applied in particular in the theory of general relativity. The subjects of elasticity are developed in plane and solid Euclidean space. Therefore we are only interested in the tensor calculus of Euclidean spaces, where it is possible to introduce also Cartesian co-ordinate systems.

For the fundamentals of the tensor calculus we shall do reference to [8] and, in particular for the reference system and the symbol choice, to [1] and [2]. Thus we shall suppose that the tensor calculus is well known to the reader, discussing only some concepts on the Cristoffel symbol and their use as a way for the expression of bending and twisting of the reference system. As the application of the tensor calculus to structures is concerned, we will discuss the adaptation to our purposes of the concepts of strain tensors, stress tensors, displacement tensors, equilibrium equations and stress–strain relations.

2.2. Strain Tensor

Let us consider the deformation of elastic media without making the usual approximations of the classical ("infinitesimal") theory. Let a three–dimensional medium in an Euclidean space be subjected to deformation from the initial (unstrained) to its final (strained) position, described by a strain tensor field. If (${}^1a$, ${}^2a$, ${}^3a$) are the curvilinear co-ordinates in an elastic medium, and $\left(x^1, x^2, x^3\right)$ are the curvilinear co-ordinates after deformation, the deformation itself is in general given by differentiable functions, 2:

$$x^i = f^i\left({}^1a, {}^2a, {}^3a\right). \tag{2.1}$$

f the initial squared element ds_0^2 and the final one ds^2 of arc length are respectively, [2]:

$$\begin{aligned} ds_0^2 &= {}_{\alpha\beta}c\left(a\right)\left(d^{\alpha}a\right)\left(d^{\beta}a\right), \\ ds^2 &= g_{\alpha\beta}\left(x\right)dx^{\alpha}dx^{\beta} \end{aligned} \tag{2.2}$$

we can also write, [2]:

$$ds_0^2 = h_{\sigma\tau} dx^\sigma dx^\tau , \qquad ds^2 = {}_{pq}k\left(d^{\,p}a\right)\left(d^{\,q}a\right), \tag{2.3}$$

Where

$$h_{\sigma\tau} = {}_{\alpha\beta}c\left({}^{\alpha}a,_{\sigma}\right)\left({}^{\beta}a,_{\tau}\right), \qquad {}_{pq}k = g_{\alpha\beta}\left({}_{p}\,,x^{\alpha}\right)\left({}_{q}\,,x^{\beta}\right). \tag{2.4}$$

With

$$\varepsilon_{\alpha\beta}\left(x\right) = \frac{1}{2}\left(g_{\alpha\beta}\right)\left(x\right) - h_{\alpha\beta}\left(x\right) \tag{2.5}$$

$${}_{\alpha\beta}\eta\left(a\right) = \frac{1}{2}\left({}_{\alpha\beta}k\left(a\right) - {}_{\alpha\beta}c\left(a\right)\right) \tag{2.6}$$

we have, [2]:

$$ds^2 - ds_0^2 = 2\varepsilon_{\alpha\beta} dx^\alpha dx^\beta \tag{2.7}$$

$$ds^2 - ds_o^2 = 2\,{}_{\alpha\beta}\eta\left(a\right)\left(d^{\,\alpha}a\right)\left(d^{\,\beta}a\right) \tag{2.8}$$

To our purposes it may be useful to choose as reference system in the strained medium the transformed of the reference system in the original medium and to put for (2.1):

$${}^{i}a = x^{i} .$$

So (2.5) becomes

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(g_{\alpha\beta} - {}_{\alpha\beta}c\right) \tag{2.9}$$

and

$$g_{\alpha\beta} = {}_{\alpha\beta}c + 2\varepsilon_{\alpha\beta} , \tag{2.10}$$

where ${}_{\alpha\beta}c$ is the Euclidean metric tensor of the reference system in the medium and $2\varepsilon_{\alpha\beta}$ is the effect of the deformations at the level of strains. Obviously no conditions can be imposed from a general point of view on the choice of the (curvilinear) co-ordinate system (1a, 2a, 3a) in the elastic medium, even if one can prefer, for instance, a system that permits a simple mathematical expression of the boundary surface.

2.3 Euclidean Christoffel Symbols

The element of arc length squared in general co-ordinates takes the form:

$$ds^2 = g_{\alpha\beta}\left(x^1, x^2, x^3\right) dx^\alpha dx^\beta , \tag{2.11}$$

where $g_{\alpha\beta}$ are the components of the covariant Euclidean metric tensor and the determinant of its components is zero, [2]:

$$g = \begin{vmatrix} g_{11} \cdots g_{12} \cdots g_{13} \\ g_{21} \cdots g_{22} \cdots g_{23} \\ g_{31} \cdots g_{32} \cdots g_{33} \end{vmatrix} =\neq 0 \tag{2.12}$$

If we define, [2]

$$g^{\alpha\beta} = \frac{cofactor - of - g_{\alpha\beta} - in - g}{g}$$

the functions $g^{\alpha\beta}$ are the components of the contravariant Euclidean metric tensor with the following properties:

$$g^{\alpha\beta} = g^{\beta\alpha} , \qquad g^{\alpha\sigma} g_{\sigma\beta} = \delta^{\alpha}_{\beta} . \tag{2.13}$$

Let us now introduce the Euclidean Christoffel symbols of the second kind $\Gamma^i_{\alpha\beta}\left(x^1, x^2, x^3\right)$:

$$\Gamma^i_{\alpha\beta}\left(x^1, x^2, x^3\right) = \frac{1}{2} g^{i\sigma}\left(\frac{\partial g_{\sigma\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right) \tag{2.14}$$

Since the law of transformation of the $g_{\alpha\beta}$ and $g^{\alpha\beta}$ is known, one can calculate the law of transformation of $\Gamma^i_{\alpha\beta}\left(x^1, x^2, x^3\right)$. Under a transformation of co-ordinates $\overline{x}^i = f^i\left(x^1, x^2, x^3\right)$, $\Gamma^i_{\alpha\beta}\left(x^1, x^2, x^3\right)$ will perform a transformation. If $\overline{g}_{\alpha\beta}$ and $\overline{g}^{\alpha\beta}$ are respectively the covariant and contravariant components of the Euclidean metric tensor in the $\overline{x}^i$ co-ordinates, the Euclidean Cristoffel symbol:

$$\overline{\Gamma}^i_{\alpha\beta}\left(\overline{x}^1,\overline{x}^2,\overline{x}^3\right)=\frac{1}{2}\overline{g}^{i\sigma}\left(\frac{\partial\overline{g}_{\sigma\beta}}{\partial x^\alpha}+\frac{\partial\overline{g}_{\alpha\sigma}}{\partial x^\beta}-\frac{\partial\overline{g}_{\alpha\beta}}{\partial x^\sigma}\right) \tag{2.15}$$

is given by the following transformation law:

$$\overline{\Gamma}^i_{\alpha\beta}\left(\overline{x}^1,\overline{x}^2,\overline{x}^3\right)=\Gamma^\lambda_{\mu\nu}\left(x^1,x^2,x^3\right)\frac{\partial x^\mu}{\partial\overline{x}^\alpha}\frac{\partial x^\nu}{\partial\overline{x}^\beta}\frac{\partial\overline{x}^i}{\partial x^\lambda}+\frac{\partial^2 x^\lambda}{\partial\overline{x}^\alpha\partial\overline{x}^\beta}\frac{\partial\overline{x}^i}{\partial x^\lambda} \tag{2.16}$$

Thus the Euclidean Christoffel symbols are not tensors and the involved indexes can't be considered as tensor indexes. Thay are of fundamental importance for the introduction of the concept of covariant derivative, that in the case of a covariant vector field (or covariant tensor field of rank one) ξ_i has the form:

$$\xi_{i,\alpha}=\frac{\partial\xi_i}{\partial x_\alpha}-\Gamma^\sigma_{i\alpha}\xi_\sigma\ ,$$

and in the case of a contravariant tensor field of rank two $T^{\alpha\beta}_{,\gamma}$ has the form:

$$T^{\alpha\beta}_{,\gamma}=\frac{\partial T^{\alpha\beta}}{\partial x^\gamma}+\Gamma^\beta_{\sigma\gamma}T^{\sigma\alpha}+\Gamma^\alpha_{\sigma\gamma}T^{\beta\sigma}\ . \tag{2.17}$$

The covariant derivatives are tensors and allow us to extend the concept of partial derivative, that in the case of a scalars fields s give a covariant vector field $\frac{\partial s}{\partial x_i}$, that can be indicated as a covariant derivative $s_{/i}$. Because of the particular choice of the reference system, we obtained (2.10). Introduced in the expression of the Euclidean Christoffel symbols of the second kind, it gives:

$$\Gamma^i_{\alpha\beta}\left(x^1,x^2,x^3\right)=\frac{1}{2}\left(2\varepsilon^{i\sigma}+{}^{i\sigma}c\right)\left(\frac{\partial\left(2\varepsilon_{\alpha\beta}+{}_{\alpha\beta}c\right)}{\partial x^\alpha}+\frac{\partial\left(2\varepsilon_{\alpha\sigma}+{}_{\alpha\sigma}c\right)}{\partial x^\beta}-\frac{\partial\left(2\varepsilon_{\alpha\beta}+{}_{\alpha\beta}c\right)}{\partial x^\sigma}\right) \tag{2.18}$$

The part of Euclidean metric tensor depending on the external loads is ε and (2.18) shows that the expression of the Euclidean Christoffel symbol contains products of ε and its derivatives

Curvatures and twists that can be expressed by (2.18) are so reduced to sums of effects of initial co-ordinate system and deformations, see also [14], [15] and [17].

2.4 Displacements as unknowns of the problem

We are now in the position to introduce the displacements due to loads as unknowns of the problem. It can be obtained by expressing the fundamental tensor of the deformed medium by means of the fundamental tensor of the medium and the displacement vector $\overline{s}({}^i a)$ referred to the ${}^i a$ reference system. Reference [8], Cap. IX, § 4 demonstrates and shows the following expression for the strain tensor in function of the displacement components:

$$ {}_{ik}\varepsilon = \frac{1}{2}\left({}_{i,k}s + {}_{k,i}s + {}_{r,i}s^{\cdot r}_{\cdot ,k}s \right). \tag{2.19}$$

Taking into account the choice $x^i = {}^i a$, such expression, if we refer the displacements to the deformed curvilinear co-ordinates $s_i = s_i(x^j)$, can be modified as follows:

$$ \varepsilon_{ik} = \frac{1}{2}\left(s_{i/k} + s_{k/i} - s_{r/i}s^{r}_{/k} \right), \tag{2.20}$$

that results the strain tensor in the deformed medium, where the Euclidean metric tensor results:

$$ g_{ik} = {}_{ik}c + \left(s_{i/k} + s_{k/i} - s_{r/i}s^{r}_{/k} \right). \tag{2.21}$$

The determination of $s_i = s_i(x^j)$ in the deformed medium can be obtained by means of a passage through a Cartesian reference system in the medium ${}^i y = {}^i y({}^j a)$, where the tensor expression ${}_i\widehat{s}$ of the displacement is:

$$ {}_i\widehat{s} = {}_j s \frac{\partial^j a}{\partial^i y}. $$

The co-ordinates of the deformed point in the Cartesian system is

$$ {}^i\overline{y} = {}^i y + {}^i\widehat{s}\,, \tag{2.22}$$

where the second member is a function of the ${}^{j}a$. The (2.22) result to be the relation between between ${}^{i}\overline{y}$ and ${}^{j}a = x^{j}$. With reference to the deformed medium the displacement tensor is given by

$$s_i = {}_j\hat{s}\,\frac{\partial \overline{y}_i}{\partial x^j} \quad . \qquad (2.23)$$

It is now possible to use the (2.21) and obtain the expression of the strain tensor in the deformed medium. It is possible to demonstrate the (2.21) in another way. If P is a point of the medium and Q is the corresponding point of the deformed medium,

$$Q = P + \overline{s} \quad \text{or} \quad P = Q - \overline{s}$$

and

$$\frac{dP}{dx^i} = \frac{dQ}{dx^i} - \overline{s}_{/i} .$$

The Euclidean metric tensors in the deformed medium and in elastic medium respectively can be expressed by means of the formulas

$$g_{ik} = \frac{dQ}{dx^i} \times \frac{dQ}{dx^k} \quad \text{and} \quad {}_{ik}c = \frac{\partial P}{\partial x^i} \times \frac{\partial P}{\partial x^k} .$$

$$\frac{\partial P}{\partial x^i} \times \frac{\partial P}{\partial x^k} = \left(\frac{dQ}{dx^i} - \overline{s}_{/i}\right) \times \left(\frac{dQ}{dx^k} - \overline{s}_{/k}\right)$$

$${}_{ik}c = g_{ik} - s_{i,k} - s_{k,i} + s_{r,i} s^{r}_{,k} ,$$

that coincides with the (2.21). Taking into account that Taking into account that $\varepsilon_{ik} = \frac{1}{2}\left(g_{ik} - {}_{ik}a\right)$, the (2.20) is obtained.

2.5. Stress Tensor

A body of elastic medium in its strained position will occupy a volume V with a bounding surface S . Let dS be the vector magnitude of a bounding surface element of a portion of the elastic medium in its strained position and dS_α its tensor representation. It is, [8],

$$(dS)^2 = g^{\alpha\beta} dS_\alpha dS_\beta .$$

If u,v are surface parameters so that the parametric equations of the surface S are given by $x^i = f^i(u,v)$, the covariant vector dS_r can be obtained by means of the mixed product of vectors

$$dS_r = \left(\frac{\partial P}{\partial x^i}\frac{\partial x^i}{\partial u} \wedge \frac{\partial P}{\partial x^j}\frac{\partial x^j}{\partial v} \right) \times \frac{\partial P}{\partial x^r} \partial u \partial v \qquad (2.24)$$

In the (2.24) there appear vector mixed products that are values different from zero only if $i \neq j \neq r$. In such a case each of them has the scalar density value $\sqrt{g}$ – see [8], Cap. II, § 9, p. 82 – which can be considered a common factor. The other factors can be obtained as addends of the development by rows of a determinant. In fact if $\overline{u}_i$ are the versors of the curvilinear co-ordinates, they are given by the formula

$$\frac{dS_r}{\sqrt{g}} = \left(\overline{u}_i \frac{\partial x^i}{\partial u} \wedge \overline{u}_j \frac{\partial x^j}{\partial u} \right) \times \overline{u}_r \partial u \partial v . \qquad (2.25)$$

In orthogonal Cartesian co-ordinates they can be obtained as addends of the development by rows, starting from the first one, of the determinant

$$\begin{vmatrix} ...1, ...1, ...1 \\ \frac{\partial y^1}{\partial u}, \frac{\partial y^2}{\partial u}, \frac{\partial y^3}{\partial u} \\ \frac{\partial y^1}{\partial v}, \frac{\partial y^2}{\partial v}, \frac{\partial y^3}{\partial v} \end{vmatrix} \partial u \partial v . \qquad (2.26)$$

In curvilinear co-ordinates the tensor dS_r is given by:

$$\begin{cases} dS_1 = \sqrt{g}\, d(x^2, x^3), \\ dS_2 = \sqrt{g}\, d(x^3, x^1), \\ dS_3 = \sqrt{g}\, d(x^1, x^2). \end{cases} \tag{2.27}$$

where

$$d\left(x^p, x^q\right) = \begin{vmatrix} \dfrac{dx^p}{du}, \dfrac{dx^p}{dv} \\ \dfrac{dx^q}{du}, \dfrac{dx^q}{dv} \end{vmatrix} dudv \tag{2.28}$$

The surface element dS can be expressed as follows:

$$(dS)^2 = g^{\alpha\beta} dS_\alpha dS_\beta .$$

The notion of stress tensor that we must introduce is based on the definition of the stress vector, i.e. a surface force that acts on the surface of a volume. A stress tensor $T^{r\alpha}$ is defined implicitly by the relation

$$F^r dS = T^{r\alpha} dS_\alpha , \tag{2.29}$$

where F^r is the stress vector acting on the surface element dS.

Beside the surface forces are to be considered the so called mass forces, i.e. forces that act throughout the volume (called body forces, volume forces and so on) An example is the force of gravity, $\rho g \Delta v$, where g is the gravitational acceleration, ρ is the medium density and Δv is the elementary volume interested.

3. Equilibrium equations

The equilibrium equation in tensor notations can be obtained from an application of the virtual work principle. We shall utilise the Lagrange's formulation of such principle, where rigid bodies are considered, even if we are interested to elastic bodies. Our deductions will be correct if we shall reference only to rigid body deformations. The principle states that, if rigid body virtual displacements are considered, the virtual work of all the external forces acting on any portion of the medium is zero, as a consequence of the zero value of the boundary reaction virtual work. This statement has also the value of physical assumption of the equilibrium. In our problems the stresses across the boundary S and the mass forces (M^r per unit mass) acting on the medium are the external forces.

As mass forces are considered, we shall do reference to its tensor representation in the deformed medium curvilinear co-ordinate system. Let us consider in the elastic medium a Cartesian coordinate system $y_i = y_i\left({}_j a\right)$. The tensor $\tilde{s}_i = \tilde{s}_i\left({}_i a\right)$ representing the displacements in such a system can be easily obtained by means of the co-ordinate transformation. In the same Cartesian system the deformed point is given by $\overline{y}_i = y_j + \tilde{s}_j$. A representation of the same point in the deformed medium can be obtained observing that we have a relation $\overline{y}_i = \overline{y}_j\left({}_i a\right) = \overline{y}_j\left(x_i\right)$, which permit the transformation of the mass force tensor referred to x_i. Obviously it is possible to consider also the case where the mass force tensor depends on y_j. The use of distributions, see also [18], allows us to extend this definition also to forces distributed on two or one dimensions.

Let us now briefly recall some elementary concepts about virtual displacements and several their consequences as virtual work and so on. A system of virtual displacements

$$\hat{x}\left({}^s a, t\right)$$

is a system coherent with the constraints that is thanked to be superimposed to the (equilibrium) configuration $x^i = x^i\left({}_s a\right)$, so obtaining a configuration

$$\overline{x}^i = x^i\left({}^s a\right) + \hat{x}^i\left({}^s a, t\right) = \overline{x}^i\left({}^s a, t\right),$$

in general not in equilibrium with the applied loads. The parameter $t \to 0$ determines the amount of virtual displacement. It is now possible to consider partial derivatives for respect to t and the corresponding variations. If $f\left(\overline{x}^i\right)$ is a tensor field, its variation can be expressed as:

$$\delta f = \left[f\left(\overline{x}^i\right)\right]_{,j} \frac{\partial \hat{x}^j}{\partial t} dt , \tag{3.1}$$

where the covariant derivative reduces to the conventional if x^i is a Cartesian reference system or $f\left(\overline{x}^i\right)$ is a scalar field.

The virtual work δL of the external forces for arbitrary $\delta x_\alpha \; \delta x_\beta$ is:

$$\delta L = \iiint_V \rho M^\beta \delta x_\beta dV + \iint_S F^\beta \delta x_\beta dS \tag{3.2}$$

where ρ is the mass density.

The second addend of the first term, for arbitrary continuous $\delta x_\alpha \; \delta x_\beta$ can be transformed as follows:

$$\iint_S T^{\beta\alpha} \delta x_\beta dS_\alpha = \iiint_V \left(T^{\beta\alpha} \delta x_\beta \right)_{,\alpha} dV ,$$

where the equivalence from the first to the second member is obtained by means of Green's theorem or generalized Stokes' theorem in curvilinear coordinates or as an application of the divergence theorem, [8] Cap. IV, § 4, n. 8, in consideration of the fact that $\left(T^{\beta\alpha} \delta x_\beta \right)_{,\alpha}$ is the divergence of a simple tensor.

Hence, for the virtual work of all the forces acting on any portion of the medium, for continuous $\delta x_\alpha \; \delta x_\beta$ operating we obtain:

$$\delta L = \iiint_V \left[\left(T^{\beta\alpha}{}_{,\alpha} + \rho M^\beta \right) \delta x_\beta + T^{\beta\alpha} \left(\delta x_\beta \right)_{,\alpha} \right] dV \tag{3.3}$$

If we consider, as rigid virtual displacements the translations, characterized by $\left(\delta x_p \right)_{,q} = 0$, we must reach the condition for rigid virtual displacements:

$$\iiint_v \left[\left(T^{\beta\alpha}_{,\alpha} + \rho M^\beta \right) \delta x_\beta dV \right] = 0 \tag{3.4}$$

Because of the arbitrariness of δx_β at any chosen point and of the portion V of the body, (3.4) impose the following equations:

$$T^{\beta\alpha}_{,\alpha} + \rho M^\beta = 0\,, \tag{3.5}$$

that are the differential equations for equilibrium.

These equations can be written directly, assuming the principle of equilibrium as the starting point of the mechanic, see [0], Cap. IX, § 4, n.7. In such a case the principle of virtual works can be deduced as a consequence, 0. The first member of (3.2) is obviously the expression of a generic virtual work for elastic displacements. So the virtual work δL of all the external forces (mass as well as surface) acting upon any portion of the medium in any virtual displacement results (on using [3.3]) and [3.5]):

$$\delta L = \iiint_V T^{\beta\alpha} \left(\delta x_\beta\right)_{,\alpha} dV \tag{3.6}$$

that can be seen as the virtual work of the internal forces. This could be a simple way to introduce the virtual work principle for the elastic media, where the virtual work of the external forces is equal to the virtual work of the internal forces, for continuous δx_α:

$$\iiint_V \rho M^\beta \delta x_\beta dV + \iint_S F^\beta \delta x_\beta dS = \iiint_V T^{\beta\alpha} \left(\delta x_\beta\right)_{,\alpha} dV\,. \tag{3.7}$$

The stress tensor results to be symmetric:

$$T^{\alpha\beta} = T^{\beta\alpha}\,, \tag{3.8}$$

because (3.6) must vanish for any rigid virtual displacement, i.e. for $\left(\delta x_q\right)_{,p} + \left(\delta x_p\right)_{,q}$.

4. Stress–strain relations

The principle of conservation of mass in presence of a virtual displacement can be written:

$$\delta(dm) = \delta(\rho\, dV) = 0, \tag{4.1}$$

where ρ is the density of the volume element dV in the strained medium and $dm = \rho \ dV$ is the element of mass.

If T is the temperature of dm and σ is the entropy density (per unit mass) the entropy of the mass *dm* is $\sigma dm = \rho\sigma dV$. If and *udm* is the internal energy of *dm,* because of the fundamental energy–conservation law of thermodynamics we have:

$$\delta\left(udm\right) - T\delta(\sigma dm) = \delta L \tag{4.2}$$

where δL is the virtual work of all the forces acting on *dm*.

The free energy or elastic potential ϕ can be written

$$\phi = u - T\sigma \tag{4.3}$$

From (4.1) we have, taking into account also (3.6), (4.2), and (4.3)

$$\iiint_V (\delta\phi)\rho dV = \iiint_V T^{\alpha\beta}(\delta x_\alpha),_\beta \, dV - \iiint_V (\delta T)\rho\sigma dV , \tag{4.4}$$

where the integrals are extended over any portion of the strained medium.

Thus we have also:

$$\rho\delta\phi = T^{\alpha\beta}(\delta x_\alpha),_\beta - \rho\sigma\delta T \tag{4.5}$$

Euclidean metric tensor $g_{ij}(x)$ in the strained medium, the Euclidean metric tensor $_{\alpha\beta}c(a)$ in the unstrained medium. [2].

A rigid virtual displacement is defined by the equations:

$$\delta\left(ds\right)^2 = \delta\left(g_{rs}dx^r dx^s\right) = \delta g_{rs}dx^r dx^s + g_{rs}\delta\left(dx^r\right)dx^s + g_{rs}dx^r\delta\left(dx^s\right) = 0. \tag{4.6}$$

The variation of the fundamental tensor is zero, $\delta g_{rs} = 0$, in any reference system, as a consequence of (3.1) and of the fact that in any reference system $g_{rs,i} = 0$ (see [8], Cap. V). The variation $\delta\left(dx^k\right)$ can be calculated as follows:

$$\delta\left(dx^k\right)=\delta\left({}_{s,}x^k\right)d^s a=\frac{\partial\left(\delta x^k\right)}{\partial^s a}d^s a=\frac{\partial\left(\delta x^k\right)}{\partial x^\alpha}\left({}_{s,}x^\alpha\right)dx^\alpha .$$

Hence we have:

$$\delta\left(dx^k\right)=\left(\delta x^k\right)_{,\alpha}dx^\alpha . \tag{4.7}$$

In conclusion, with some calculations we can obtain the equation

$$\left(\delta x_\alpha\right)_{,\beta}+\left(\delta x_\beta\right)_{,\alpha}=0 . \tag{4.8}$$

From the symmetry of the stress tensor, and (4.8), it results that in an isothermal rigid virtual displacement the second member of (4.5) gives $\delta\phi=0$. Hence

$$\frac{\partial\phi}{\partial\left({}^\alpha a_{,\beta}\right)}\delta\left({}^\alpha a_{,\beta}\right)=0 \tag{4.9}$$

Taking into account that $\delta\left({}^\alpha a_{,\beta}\right)=-{}^r a_{,\alpha}\left(\delta x^\alpha\right)_{,\beta}$ and, from (4.8) we obtain that if ${}^r a^\alpha\left(x\right)=g^{\alpha\beta}\left(x\right){}^r a_{,\beta}$, ϕ must satisfy the following complete system of three linear first–order partial differential equations in the nine variables ${}^\alpha a_{,\beta}$:

$$\frac{\partial\phi}{\partial\left({}^\alpha a_{,\beta}\right)}{}^\alpha a^\gamma=\frac{\partial\phi}{\partial\left({}^\alpha a_{,\gamma}\right)}{}^\alpha a^\beta . \tag{4.10}$$

There are nine conditions in (4.10) but three are identities and only three of the remaining six are independent. From the theory of such systems of differential equations we know that the general solution of (4.10) is a function of six functionally independent solutions.

There are some interesting solutions of equations (4.10), in particular in the case of an isotropic medium, i.e. a medium whose elastic potential is a strain invariant that may depend parametrically on the temperature T.

The elastic potential for an isotropic medium satisfies the differential equations (4.10), see [1]. It can also be shown that any strain invariant is a function of the three fundamental strain invariants $I_{1,}I_2$ and I_3, that are functions of the strain tensor ε_s^r, see [1] § [3] and appendix. Hence for an isotropic medium

$$\phi = \phi\left(I_1, I_2, I_3, T\right),$$

i.e. ϕ is a function of the strain tensor ε_s^r and T.

Conversely, from the assumption that ϕ is a function of the strain tensor ε_s^r and T the following result is obtained: ϕ is isotropic; see [1], § [3].

4.1 Stress–strain Relations for an Isotropic Medium

As we have just shown, the elastic potential ϕ for an isotropic medium can be considered as a function of the strain tensor ε_{rs}. Because of the symmetry of ε_{rs}, we have also $\varepsilon_{rs} = \frac{1}{2}\left(\varepsilon_{rs} + \varepsilon_{sr}\right)$. In ϕ, we shall write $\frac{1}{2}\left(\varepsilon_{rs} + \varepsilon_{sr}\right)$ wherever ε_{rs} occurs, and thus we see that

$$\frac{\partial \phi}{\partial \varepsilon_{rs}} = \frac{\partial \phi}{\partial \varepsilon_{sr}} \tag{4.11}$$

with the understanding that in $\frac{\partial \phi}{\partial \varepsilon_{rs}}$, say, all the other ε's (including ε_{sr} for that *s,* r) are held constant, so that in this differentiation no attention is paid to the symmetry relations $\varepsilon_{rs} = \varepsilon_{sr}$.

Under a virtual displacement the variation of h_{pq} and hence of the strain tensor ε_{pq} was given by

$$\delta\varepsilon_{pq} = -\frac{1}{2}\delta h_{pq} = \frac{1}{2}\left[h_q^\tau\left(\delta x_\tau\right)_{,p} + h_p^\tau\left(\delta x_\tau\right)_{,q}\right] \tag{4.12}$$

since $h_{pq} = g_{pq} - 2\varepsilon_{pq}$. But $\delta g_{pq} = 0$; hence from (4.9) we otain:

$$\delta\phi = \frac{\partial \phi}{\partial \varepsilon_{\alpha\beta}}\delta\varepsilon_{\alpha\beta} = \frac{1}{2}\frac{\partial \phi}{\partial \varepsilon_{\alpha\beta}}\left[h_q^\tau\left(\delta x_\tau\right)_{,p} + h_p^\tau\left(\delta x_\tau\right)_{,q}\right],$$

and taking into account also (4.12):

$$\delta\phi = \frac{\partial\phi}{\partial\varepsilon_{\alpha\beta}} h_\beta^\tau \left(\delta x_r\right),_\alpha \tag{4.13}$$

From (4.5) in the case of an isothermal virtual displacement, and an isotropic medium it is possible to obtain:

$$\rho \frac{\partial\phi}{\partial\varepsilon_{\alpha\beta}} h_\beta^\tau \left(\delta x_r\right),_\alpha = T^{\alpha\beta} \left(\delta x_\alpha\right),_\beta . \tag{4.14}$$

A virtual displacement is arbitrary and the relation $h_q^p = \delta_q^p - 2\varepsilon_q^p$ is valid. So (4.14) means that is valid the following stress–strain relation: we obtain the stress–strain relations for an isotropic medium

$$T^{\alpha\beta} = \rho\left(\frac{\partial\phi}{\partial\varepsilon_{\beta\alpha}} - 2\varepsilon_\sigma^\alpha \frac{\partial\phi}{\partial\varepsilon_{\beta\sigma}}\right) \tag{4.15}$$

Taking into account the relations between covariant and mixed tensor the (4.15) can be reduced to the form

$$T_\beta^\alpha = \rho\left(\frac{\partial\phi}{\partial\varepsilon_\alpha^\beta} - 2\varepsilon_\sigma^\alpha \frac{\partial\phi}{\partial\varepsilon_\sigma^\beta}\right) \tag{4.16}$$

Since I_1, I_2, and I_3 are respectively first degree, second degree, and third degree in the strain tensor components $\varepsilon_{\alpha\beta}$, see [1]§ [3], to a first approximation the stress–strain relations (4.16) for an isotropic medium reduce to Hooke's law of the usual approximate theory

$$T_\beta^\alpha = \frac{\partial\rho\phi^1}{\partial\varepsilon_\alpha^\beta} . \tag{4.17}$$

The hypothesis that the elastic potential is a function of the derivatives ${}^r a,_s$, of the Euclidean metric tensor $g_{ij}(x)$ in the strained medium, of the Euclidean metric tensor ${}_{\alpha\beta}c(a)$ in the unstrained medium, and of the temperature T allows us to reach the conclusion that at a constant temperature both eq. (4.16) and eq.(4.17) give the stress tensor as a function of the strain tensor . Taking into account eq. (2.20), we can conclude that the stress tensor is a function of the covariant derivative of the displacement s_i . In ref. [8], Cap. IX, § [5], with reference to the case of small strains – eq. (4.17) – the following relation is indicated

$$T_{ik} = c_{ikrs}\varepsilon^{rs},$$

where the elastic tensor c_{ikrs} appears. The quoted reference takes into account the unstrained medium and the proper reference system; because of the particular choice of the reference system in the strained medium we can adopt there the same formulation. The elastic tensor results to be symmetric with respect to the two first and the two last indices. For an isotropic medium the elastic tensor can be put under the form:

$$c_{ikrs} = -\lambda x_{ik} x_{rs} - \mu\left(x_{ir} x_{ks} + x_{kr} x_{is}\right),$$

where λ and μ are the Lame's constants and x_{ik} is Euclidean metric tensor in the medium. In [1] one can find the demonstration that, in order to approximate the expansion by series of the elastic potential up to the first order, two constant are necessary – as μ, ν – and to reach the second order five constants are necessary.

4.2. Non–linearities in the equilibrium equations

The equilibrium equation (3.5), beside the mass forces, contains the covariant derivative of the stress tensor. We can put the problem under the form of the following equation system:

$$\left\{T^{\beta\alpha}_{/\alpha} + \rho M^{\beta} = 0\right.,$$

$$,\left\{T^{\beta\alpha}_{/k} = \frac{\partial T^{\beta\alpha}}{\partial x^k} - T^{\sigma\alpha}\Gamma^{\alpha}_{\sigma k} - T^{\beta\sigma}\Gamma^{\alpha}_{\sigma k}\right.$$

$$T_{ik} = c_{ikrs}\varepsilon^{rs},$$

$$g_{\alpha\beta} = {}_{\alpha\beta}c + 2\varepsilon_{\alpha\beta}, \tag{4.18}$$

$$\varepsilon_{ik} = \left(s_{i/k} + s_{k/i} - s_{r/i}s^{r}_{/k}\right),$$

$$s_{i/\alpha} = \frac{\partial s_i}{\partial x^{\alpha}} - \Gamma^{\sigma}_{i\alpha}s_{\sigma},$$

where the unknowns $T^{\alpha\beta}, g_{\alpha\beta}, \varepsilon_{\alpha\beta}, s_i$ – functions of x_i – appear beside the known quantities $_{\alpha\beta}c, c_{ikrs}$ and where the Christoffel symbols – that are not tensors – are given by (2.18). The (4.18) system and the (2.18) can be considered a system where the unknowns are the s_i. Such system contains all the non–linear component of the problem. Obviously simplifications of the mathematical model can be obtained, so reducing it to approximated analyses, disregarding terms of various order in the s_i unknowns and theirs derivatives. Beside the assumption that the ε are small, a rationale way of thinking, based on the mathematical model itself, in order to choose the terms to be disregarded is impossible. The choice must be operated for each problem, taking into account, beside the field equations, the boundary conditions, because they can strongly influence the amount of the response to the various terms.

When approximated approaches are accepted, other simplifications are possible, as for instance to determine the response to external loads $M^{\beta} - \varphi^{\beta}$, where φ^{β} allows us to take in consideration a problem that, by means of a proper mathematical model, admits as an easy resolution a configuration $s*$, and to assume such configuration $s*$ as reference for a simplified mathematical model where φ^{β} are the external loads and $\tilde{s}$ are the unknowns – for instance a linear mathematical model taking into account curvatures and twists of $s*$ and disregarding those due to $\tilde{s}$.

Sample problem

In order to give a simple applied example of approximated approach, let us consider an hollow initially curved beam with a thin walled square orthogonal section of side b, subjected to an external force system F consisting in a constant bending moment parallel to a section symmetry axe – say the $x \equiv x^1$ axe. The c.g. initial axe has a curvature ray r_0 in the plane perpendicular to the x^1 axe, that results a symmetry plane for the beam. The c.g. deformed axe will be contained in the symmetry plane, where the axes x^2 – contained in the normal section – and x^3 – orthogonal to the normal section – are posed. In a first approximation sufficient for our purposes, the c.g. axe will assume the form of an arch of circle o R_0 ray. In a real behaviour the square section will assume a form not still square – in particular both the compressed and the stretched walls will assume a displacement toward the inside of the square. Such kind of deformation can be avoided in a first degree of analysis introducing a distribution ψ of external mass forces applied to the two interested walls in such a manner that we can consider that in a first approximation the square section will assume a form still square. The Euclidean metric tensor of the medium reference system has only three components different from zero: ${}_{11}c = 1$, ${}_{22}c = 1$ and ${}_{33}c = \left(\frac{r_0 - x^2}{r_0} \right)^2 = \frac{r^2}{r_0^2}$, where $r = r\left(x^2\right)$ is the radius of curvature at the x^2 position. Thus it is possible to calculate the deformation by means of the ordinary beam theory. If the Poisson modulus is considered zero, the deformation under the $F - \psi$ external load system creates a strain tensor with only $\varepsilon_{33} = \frac{1}{2}\left(\frac{R_0 - x^2}{R_o} \right)^2 - \frac{1}{2}\left(\frac{r_0 - x^2}{r_0} \right)^2$ is different from zero and a stress tensor with only $T_{33} = E\varepsilon_{33}$ different from zero, where $R = R(y)$ is the radius of curvature at the x^2 position and E is the elastic modulus.

Taking into account (2.21) and (2.20), in the Euclidean metric tensor in the strained medium only $g_{11} = 1$, $g_{22} = 1$ and $g_{33} = \left(\frac{R_o - x^2}{R_0} \right)^2$ result different from zero. Taking into account (2.15), the Cristoffel symbol gives $\Gamma_{33}^2 = \frac{1}{\left(R_0 - x^2\right)} = \frac{1}{R}$.

In the equilibrium equation (3.5), *M* being equal to zero, the not summed index indicate the equilibrium component. From (2.17) it is possible to obtain

$$T^{\beta\alpha}_{,\alpha} = 0$$

and

$$\begin{aligned} T^{\beta\alpha}_{,\alpha} &= \frac{\partial T^{\beta 1}}{\partial x^1} + \frac{\partial T^{\beta 2}}{\partial x_2} + \frac{\partial T^{\beta 3}}{\partial x^3} + \\ &+\Gamma^{\beta}_{11} T^{11} + \Gamma^{\beta}_{12} T^{12} + \Gamma^{\beta}_{13} T^{13} + \Gamma^{\beta}_{21} T^{21} + \Gamma^{\beta}_{22} + \Gamma^{\beta}_{23} T^{23} + \Gamma^{\beta}_{31} T_{31} + \Gamma^{\beta}_{32} T^{32} + \Gamma^{\beta}_{33} T^{33} + \\ &+\Gamma^{1}_{11} T^{\beta 1} + \Gamma^{2}_{12} T^{\beta 1} + \Gamma^{3}_{13} T^{\beta 3} + \Gamma^{1}_{21} T^{\beta 2} + \Gamma^{2}_{22} T^{\beta 2} + \Gamma^{3}_{23} T^{\beta 2} + \Gamma^{1}_{31} T^{\beta 3} + \Gamma^{2}_{32} T^{\beta 3} + \Gamma^{3}_{33} T^{\beta 3} \end{aligned}$$

The addend of the second member that contain Cristoffel symbols are different from zero only if thay contain T^{33}. If we limit our analysis only to the equilibrium equation $\beta = 2$ we obtain:

$$\frac{\partial T^{21}}{\partial x^1}+\frac{\partial T2^{22}}{\partial x_2}+\frac{\partial T^{23}}{\partial x^3}+\Gamma^2_{33}T^{33}=0$$

The term $T^{33}\Gamma^2_{33}$ is a contribution deriving from the combination between the stress T^{33} and the effects of the deformation of the reference system, that is $\Gamma^2_{33}=\frac{1}{R}$. The force distribution ψ results

$$\psi = T^{33}\frac{1}{R}.$$

Let us now assume the form due to the application of the force system $F-\psi$, with the curvilinear co-ordinate system x^i obtained as deformation of the initial system ${}^j a$, as initial con figuration for the analysis of the effects of ψ, in the system x^i itself. A displacement system S_i generates effects that can be calculated in order to verify that the various axiomatic theories can be obtained with simplifications of the tensor one. For instance, a displacement system where only the S_2 is different from zero generates a well known extension term in the x^3 direction. In fact equation (2.19) gives

$$_{ik}\varepsilon=\frac{1}{2}\left({}_{i,k}S+{}_{k,i}S+{}_{r,i}S^{\cdot r}_{\cdot,k}S\right)$$

and calculating we obtain

$$\varepsilon_{33}=\frac{S_2}{R}.$$

5. Boundary conditions

The mathematical model, beside the field equations, is completed by the boundary conditions. They are surface forces applied to part of the boundary surface and displacement imposed (constraints) to the remaining part. Obviously such conditions are to be put on the deformed medium boundary surface, but this doesn't generate problems due to the combination between deformations and forces.

6. Structural stability analysis

Static stability analysis, under conservative external loads, for infinitesimal displacements δS_i, can be performed utilising the deformed curvilinear co-ordinate system as initial system and determining the lowest value of P_{cr} and the associated infinitesimal displacement system, where the first variation of the external actions, necessary in order to have the equilibrium, is zero. Because the δS_i are infinitesimal, the equilibrium equations can be reduced to a linear form as the δS_i are concerned. Obviously the analysis can be performed also determining the lowest value of P_{cr} and the associated infinitesimal displacement system, where the second variation of the total potential energy is zero. Where the equilibrium equations are reduced to linear form, the problem can be resolved also by means of the total potential properties, [9] and [10]. The existence of the involved eigensolutions is demonstrated, [13].

Conclusions

The problems of elasticity, where non–linearity derives from geometrical effects of combination between deformations and stresses, have been studied with a tensor three–dimensional approach. Deformation effects have been analysed by means of the Cristoffel symbols in the deformed medium with respect to that of the elastic medium, in a Lagrangean approach to the structural behaviour, where displacements are chosen as unknowns of the problem. The non–linearity in the equilibrium equations was completely taken into account in a proposed equation system.

An equation system, where the unknowns are the displacements due to the applied forces, was obtained, that allows us to take into account all the non linearities generated by the combination between stresses and deformations. The choice among the terms to be disregarded must be operated for each problem.

When approximated approaches are accepted, by means of proper mathematical models the configuration under an external load system with a no great difference from the applied one, can be assumed as reference for a simplified linear mathematical model to apply the difference between the external load systems.

A simple application to an hollow initially curved beam with a thin walled square orthogonal section, subjected to a constant bending moment, allowed us to show the presence of a crushing pressure on both the compressed and the stretched wall and to evaluate the amount of such a pressure, deriving from a combination between stresses and deformations, due to the bending moment.

The interaction between deformations and stresses is particularly important in those structures where components with small thickness and wide extensions are present.

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